6. Operator-valued free central limit theorem and semicircular elements

6.1. Remarks: 1) A central limit theorem asks about the limit distribution

$$\underbrace{D_{1/\sqrt{N}}}_{\text{dilation by } \frac{1}{\sqrt{N}}}(\underbrace{\mu \boxplus \dots \boxplus \mu}_{N \text{ times}}) \xrightarrow{N \to \infty} \; ?$$

or in terms of random variables

$$\frac{X_1 + \dots + X_N}{\sqrt{N}} \xrightarrow{N \to \infty} \; ?$$

if X_i are free and idendically distributed (f.i.d.)

The relevant convergence is "in distribution", which means that moments converge; since moments are elements in $\mathcal{B}$, we also have to specify the type of convergence there – we will usually take convergence in norm in $\mathcal{B}$.

2) The relevant information about the input distribution is the second moment (first moments are assumed to be zero); in the operator-valued case the second moment is given by a mapping $\eta : \mathcal{B} \to \mathcal{B}$ with

$\eta(b) := E[XbX]$

In a C^*-setting E, and thus also η, must be completely positive:

for $(b_{ij})_{i,j=1}^n \in M_n(\mathcal{B})$ we have

$$id \otimes E\big[\underbrace{\underbrace{1 \otimes X}_{\begin{pmatrix} X & & 0 \\ & \ddots & \\ 0 & & X \end{pmatrix}} \cdot (b_{ij}) \cdot \underbrace{1 \otimes X}_{\begin{pmatrix} X & & 0 \\ & \ddots & \\ 0 & & X \end{pmatrix}}}_{(X b_{ij} X)_{i,j=1}^n}\big] =$$

$$= \big(\underbrace{E(X b_{ij} X)}_{\eta(b_{ij})}\big)_{i,j=1}^n$$

$$= id \otimes \eta\,((b_{ij})_{i,j=1}^n)$$

thus $id \otimes \eta\,(bb^*) = id \otimes E[(1 \otimes X \cdot b)(1 \otimes X \cdot b)^*]$
$$\geq 0$$

We also have that every completely positive η can show up as second moment of a $\mu \in \Sigma_{\mathcal{B}}^{0}$.

($\rightarrow$ Assignments)

3) Much of the calculations for the CLT and description of the limit is similar to the scalar-valued situation (see Section 2 of FPT class). Let us first check the calculation of the moments in the limit.

We consider $(X_i)_{i \in \mathbb{N}}$ which are f.i.d. with respect to $E : \mathcal{A} \to \mathcal{B}$. We also assume that

- X_i centred : $E[X_i] = 0 \quad \forall i \in \mathbb{N}$
- second moment is given by $\eta : \mathcal{B} \to \mathcal{B}$,
  $E[X_i b X_i] = \eta(b) \quad \forall i \in \mathbb{N}, b \in \mathcal{B}$

Then we put

$$\frac{X_1 + \dots + X_N}{\sqrt{N}} =: S_N$$

and calculate its moments:

$$E[S_N b_1 S_N b_2 \cdots S_N b_{k-1} S_N] =$$

$$= \frac{1}{N^{k/2}} \sum_{i:[k]\to[N]} E[X_{i(1)} b_1 X_{i(2)} \cdots X_{i(k-1)} b_{k-1} X_{i(k)}]$$

$$= \frac{1}{N^{k/2}} \sum_{\pi \in \mathcal{P}(k)} \sum_{\substack{i:[k]\to[N] \\ \ker i = \pi}} \underbrace{E[X_{i(1)} b_1 X_{i(2)} \cdots b_{k-1} X_{i(k)}]}_{=: g(\pi)}$$

depends only on $\ker i$ by 5.2.

$$= \frac{1}{N^{k/2}} \sum_{\pi \in \mathcal{P}(k)} g(\pi) \cdot \underbrace{\#\{i:[k]\to[N] \mid \ker i = \pi\}}_{\sim N^{\#\pi}}$$

then observe: π has singleton $\Rightarrow$ $g(\pi) = 0$

because: $E[X_i] = 0$ and factorization

$$E[a_1 a_2 \tilde{a}_1] = E[a_1 E[a_2] \tilde{a}_1]$$

$\Rightarrow$ only $\pi \in \mathcal{P}_k$ without singleton contribute, i.e.

$\#\pi \le \frac{k}{2}$

in the limit $N \to \infty$ only π with $\#\pi = \frac{k}{2}$ survive; those are pairings $\pi \in \mathcal{P}_2(k)$

if π is crossing, then definition of freeness (and interval stripping) gives

$g(\pi) = 0$; e.g.,

$$g(\,) = E[X_1 b_1 X_2 b_2 X_1 b_3 X_3 b_4 X_3 b_5 X_2]$$

$$= E[\underbrace{X_1 b_1}\underbrace{X_2 b_2}\underbrace{X_1 b_3 E[X_3 b_4 X_3]}\underbrace{b_5 X_2}]$$

alternating and centred

$$= 0$$

thus: $\lim_{N\to\infty} E[S_N b_1 S_N \dots b_{k-1} S_N] = \sum_{\pi \in NC_2(k)} g(\pi)$

Now, in difference to scalar-valued case $g(\pi)$ is not the same for all $\pi \in NC_2(k)$

$$g(\sqcup) = E[X_1 b_1 X_1] = \eta(b_1)$$

$$g(\sqcup\sqcup) = E[X_1 b_1 X_1 b_2 X_2 b_3 X_2]$$

$$= E[\underbrace{E[X_1 b_1 X_1]}_{\eta(b_1)} b_2 \underbrace{E[X_2 b_3 X_2]}_{\eta(b_3)}]$$

$$= \eta(b_1) b_2 \eta(b_3)$$

$$g(\sqcup) = E[X_1 b_1 X_2 b_2 X_2 b_3 X_1]$$

$$= E[X_1 b_1 \underbrace{E[X_2 b_2 X_2]}_{\eta(b_2)} b_3 X_1]$$

$$= \eta(b_1 \eta(b_2) b_3)$$

Thus the limit variable S has moments

$$E[S b_1 S] = \eta(b_1)$$

$$E[S b_1 S b_2 S b_3 S] = \eta(b_1) b_2 \eta(b_3) + \eta(b_1 \eta(b_2) b_3)$$

$$E[S b_1 S \dots S b_{k-1} S] = \sum_{\pi \in NC_2(k)} \eta_\pi(b_1, \dots, b_{k-1})$$

where $\eta_\pi : \mathcal{B}^{k-1} \to \mathcal{B}$ is $\mathbb{C}$-multilinear map

$$\eta_\pi(b_1, \dots, b_{k-1}) = E_\pi[X_i b_1, X_i b_2, \dots, X_i b_{k-1}, X_i]$$

4) Note that even in the case where all $b_1, \dots, b_{k-1}$ are equal to 1, the contributions of the $\eta_\pi(1, 1, \dots, 1)$ are different in general:

$\eta_{\sqcup\sqcup}(1,1,1) = \eta(1)\cdot 1\cdot \eta(1) = \eta(1)^2$

$\eta_{\underline{\sqcup\!\sqcup}}(1,1,1) = \eta(1\cdot\eta(1)\cdot 1) = \eta(\eta(1))$

Note that η does not need to be unital: $\eta(1) \neq 1$ in general

<u>6.2. Theorem and Definition</u>: Let $(\mathcal{A}, \mathcal{B}, E)$ be a $\mathcal{B}$-valued C^*-probability space. Consider selfadjoint $X_i \in \mathcal{A}$ $(i \in \mathbb{N})$ which are f.i.d. (free and identically distributed) with

- $E[X_i] = 0 \quad \forall i \in \mathbb{N}$
- $E[X_i b X_i] = \eta(b) \quad \forall i \in \mathbb{N}, b \in \mathcal{B}$

for a completely positive $\eta: \mathcal{B} \to \mathcal{B}$

Put $S_N := \dfrac{X_1 + \dots + X_N}{\sqrt{N}}$

Then μ_{S_N} converges in distribution for $N \to \infty$ to $\nu_\eta \in \Sigma_{\mathcal{B}}^0$ which is given by $(k \in \mathbb{N};\ b_0, \dots, b_k \in \mathcal{B})$

$$\nu_\eta(b_0 x b_1 \cdots b_{k-1} x b_k) = \sum_{\pi \in NC_2(k)} b_0\, \eta_\pi(b_1, \dots, b_{k-1})\, b_k$$

(in particular: odd moments are zero)

$\nu_\eta \in \Sigma_B^0$ is called <u>B-valued semicircular distribution</u>, with covariance η.

<u>6.3 Remarks</u>: 1) Note that this definition is compatible with amplifications: if S is a semicircular element in $(\mathcal{A}, B, E)$ with covariance $\eta : B \to B$, then $1 \otimes S$ is a semicircular element in $(M_n(\mathcal{A}), M_n(B), id \otimes E)$ with covariance $id \otimes \eta : M_n(B) \to M_n(B)$

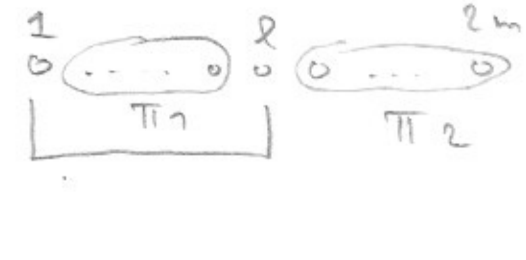
⊛ 2)

<u>6.4. Remark</u>: In order to derive an equation for the Cauchy transform of ν_η we are looking for recursions among the moments. Consider (with $\mu_S = \nu_\eta$)

$$E[Sb_1Sb_2 \cdots b_{2m-1}S] = \sum_{\pi \in NC_2(2m)} \eta_\pi(b_1, \ldots, b_{2m-1})$$

We write $\pi \in NC_2(2m)$ in the form

$\pi = (1, \ell) \cup \pi_1 \cup \pi_2$; necessarily $\ell = 2k$ even



⊛ 2) Let us check that indeed $\nu_\eta \in \Sigma_B^\circ$, i.e. that we have positivity and exponential boundedness.

One can do this by constructing operators on the full Fock space which have ν_η as distribution (→ Assignment). We do it here more abstractly

i) positivity

Since positivity is preserved in a central limit, we only need a distribution $\mu \in \Sigma_B^\circ$ which has first moment zero and second moment given by η. In Ass. 6, Ex. 1 we construct such a distribution, an operator-valued Bernoulli element.

ii) exponential boundedness

We have to estimate

$$\nu_\eta(x b_1 \ldots b_{k-1} x) = \sum_{\pi \in NC_2(k)} \eta_\pi(b_1, \ldots, b_{k-1})$$

for $k = 2m$ even

Note that η as a positive map is bounded, i.e

$$\|\eta(b)\| \le \|\eta\| \cdot \|b\| \qquad \forall\, b \in \mathcal{B}$$

and $\|\eta\| < \infty$

This implies that we have for each $\pi \in NC_2(2m)$

$$\|\eta_\pi(b_1, \dots, b_{2m-1})\| \le \|\eta\|^m \|b_1\| \cdots \|b_{2m-1}\|$$

e.g.

$$\|\eta_{\sqcup\sqcup}(b_1, b_2, b_3)\| = \|\eta(b_1)\, b_2\, \eta(b_3)\|$$

$$\le \|\eta(b_1)\| \cdot \|b_2\| \cdot \|\eta(b_3)\|$$

$$\le \|\eta\|^2 \cdot \|b_1\| \cdot \|b_2\| \cdot \|b_3\|$$

$$\|\eta_{\text{⨃}}(b_1, b_2, b_3)\| = \|\eta(b_1\, \eta(b_2)\, b_3)\|$$

$$\le \|\eta\|\ \|b_1\, \eta(b_2)\, b_3\|$$

$$\le \|\eta\|^2 \|b_1\|\ \|b_2\|\ \|b_3\|$$

Thus

$$\|\nu_\eta(x b_1 \dots b_{2m-1} x)\| \le \# NC_2(2m)\ \|\eta\|^m \|b_1\| \cdots \|b_{2m-1}\|$$

$$\le \underbrace{2^{2m} \cdot \|\eta\|^m}_{(2 \cdot \|\eta\|)^{2m}} \cdot \|b_1\| \cdots \|b_{2m-1}\|$$

3) In (1) we said that if S is B-valued semicircular, then (6-8c)

$1 \otimes S = \begin{pmatrix} S & & 0 \\ & S & \\ 0 & & S \end{pmatrix}$ is also semicircular, over

$M_m(B)$. This is true more general; if we have free semicircular elements over B and put linear combinations of them as entries into an $m \times m$-matrix, then this is an $M_m(B)$-valued semicircular element. The proof can be done by using our free central limit theorem. Let us elaborate on this via the example

$$S = \begin{pmatrix} 0 & S_1 \\ S_1 & S_2 \end{pmatrix},$$

where S_1, S_2 are free and semicircular over B, with covariances η_1, η_2.

Then we can realize S_1 and S_2 as

$$S_1 = \lim_{N \to \infty} \frac{X_1 + \dots + X_N}{\sqrt{N}}$$

$$S_2 = \lim_{N \to \infty} \frac{Y_1 + \dots + Y_N}{\sqrt{N}}$$

and

$E[X_i] = 0 = E[Y_j]$

$E[X_i b X_i] = \eta_1(b)$

$E[Y_j b Y_j] = \eta_2(b)$

where all X_i, Y_j are free

and we have w.r.t. $id \otimes E$

$$S = \lim_{N \to \infty} \begin{pmatrix} 0 & \frac{x_1 + \dots + x_N}{\sqrt{N}} \\ \frac{x_1 + \dots + x_N}{\sqrt{N}} & \frac{y_1 + \dots + y_N}{\sqrt{N}} \end{pmatrix}$$

$$= \lim_{N \to \infty} \frac{1}{\sqrt{N}} \left[\begin{pmatrix} 0 & x_1 \\ x_1 & y_1 \end{pmatrix} + \dots + \begin{pmatrix} 0 & x_N \\ x_N & y_N \end{pmatrix} \right]$$

are f.i.i.d. w.r.t. $id \otimes E$
with vanishing first moment

thus, by our central limit theorem, S is an $M_2(B)$-valued semicircular element.
It variance η is given by the second moment

$$\eta \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = id \otimes E \left[\underbrace{\begin{pmatrix} 0 & S_1 \\ S_1 & S_2 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} 0 & S_1 \\ S_1 & S_2 \end{pmatrix}} \right]$$

$$\begin{pmatrix} S_1 b_{22} S_1 & S_1 b_{21} S_1 + S_1 b_{22} S_2 \\ S_1 b_{12} S_1 + S_2 b_{22} S_1 & \begin{matrix} S_1 b_{11} S_1 + S_2 b_{21} S_1 \\ + S_1 b_{12} S_2 + S_2 b_{22} S_2 \end{matrix} \end{pmatrix}$$

$$= \begin{pmatrix} \eta_1(b_{22}) & \eta_1(b_{21}) \\ \eta_1(b_{12}) & \eta_1(b_{11}) + \eta_2(b_{22}) \end{pmatrix}$$

$$\eta_\pi(b_1, \dots, b_{2m-1}) =$$

$$= E_\pi[\,S\,\underbrace{b_1 \dots S b_{\ell-1}}_{\pi_1}\,S\,\underbrace{b_\ell S \dots b_{2m-1} S}_{\pi_2}\,]$$

$$= \eta\left(E_{\pi_1}[b_1 S \dots S b_{\ell-1}]\right) \cdot E_{\pi_2}[b_\ell S \dots b_{2m-1} S]$$

Thus $E[S b_1 \dots b_{2m-1} S] =$

$$= \sum_{k=1}^{m} \sum_{\substack{\pi_1 \in NC_2(1,\dots,2k-1) \\ \pi_2 \in NC_2(2k+1,\dots,2m)}} \eta(E_{\pi_1}[\dots]) \cdot E_{\pi_2}[\dots]$$

$$= \sum_{k=1}^{m} \eta\Big(\underbrace{\sum_{\pi_1} E_{\pi_1}[b_1 S \dots S b_{\ell-1}]}_{E[b_1 S \dots S b_{\ell-1}]}\Big) \cdot \underbrace{\sum_{\pi_2} E_{\pi_2}[\dots]}_{E[b_\ell S \dots b_{2m-1} S]}$$

$(\ell = 2k)$

and hence

$$E[S b_1 \dots S b_{2m-1} S] = \sum_{k=1}^{m} \eta\left(E[b_1 S \dots S b_{2k-1}]\right) \cdot$$

$$\cdot E[b_{2k} S \dots S_{2m-1} S]$$

Consider now the operator-valued Cauchy transform (one base level)

$$G = G_S : H^+(B) \to H^-(B)$$

$$z \mapsto E[(z-S)^{-1}]$$

For large $\|z\|$ we have

$$G(z) = z^{-1} \sum_{m \geq 0} E[(S z^{-1})^{2m}]$$

$$= z^{-1} + z^{-1} \cdot \underbrace{\sum_{m \geq 1} E[(S z^{-1})^{2m}]}$$

$$\sum_{k=1}^{m} \underbrace{\eta(z^{-1} E[(S z^{-1})^{2(m-1)}])}_{\sum_{m=1}^{\infty} \cdots = \eta(G(z))} \cdot \underbrace{z^{-1} \cdot E[(S z^{-1})^{2(m-k)}]}_{\sum_{m-k=0}^{\infty} \cdots = G(z)}$$

$$\Rightarrow z \cdot G(z) = 1 + \eta(G(z)) \cdot G(z) \qquad (*)$$

Thus: $G(z)$ satisfies for large $\|z\|$ the equation $(*)$; by analytic extension it must then satisfy $(*)$ also for all $z \in H^+(B)$.

The same calculation and arguments works also for all matricial amplifications.