Non-Commutative Distributions

0. Introduction

We are interested in (analytic) properties of distributions $\mu_{X_1,\dots,X_n}$ of

- operators $X_1,\dots,X_n$ on Hilbert spaces (typically from finite von Neumann algebras)
- often those operators are "limits in distribution" of random matrix models
- typically our operators don't commute, which makes our distributions "non-commutative"

0.1. Classical case: Consider first the classical case of "commutative" distributions. Then random variables $X_1,\dots X_n$ are measurable functions $X_i : \Omega \to \mathbb{R}$ $(i = 1,\dots,n)$ where $(\Omega, \mathcal{A}, P)$ is a probability space, i.e. P probability measure on Ω, and the distribution $\mu_{X_1,\dots,X_n}$ is a probability measure on $\mathbb{R}^n$, given as push-forward of P, i.e.

$$\mu_{X_1,\dots,X_n}(B) = P(\{\omega \in \Omega \mid (X_1(\omega),\dots,X_n(\omega)) \in B\})$$

for $B \in \mathcal{B}(\mathbb{R}^n)$

[Borel sets of $\mathbb{R}^n$

There are various ways of describing / working with this object: $\mu_{X_1,\dots,X_n}$ is

i) a probability measure on $\mathbb{R}^n$

ii) a positive linear map, which allows to average over continuous functions of $X_1,\dots,X_n$

$$E[f(X_1,\dots,X_n)] = \int_{\mathbb{R}^n} f(t_1,\dots,t_n)\, d\mu_{X_1,\dots,X_n}(t_1,\dots,t_n)$$
$$= \int_\Omega f(X_1(\omega),\dots,X_n(\omega))\, dP(\omega)$$

$f: \mathbb{R}^n \to \mathbb{C}$

[This is the same as (i) via Riesz representation theorem.]

iii) uniquely determined by its Fourier transform

$$\mathcal{F}_\mu(t_1,\dots,t_n) = E[e^{-i\langle X,t\rangle}]$$

or other nice analytic functions on $\mathbb{R}^n / \mathbb{C}^n$ [e.g. for $n=1$, the Cauchy / Stieltjes transform $G(z) = E[\frac{1}{z-X}]$]

iv) in many case (e.g., compactly supported case) uniquely determined by its moments

$$E[X_1^{k_1} \cdots X_n^{k_n}] \qquad \text{for all } k_1, \dots, k_n \in \mathbb{N}_0$$

<u>0.2. Non-commutative case:</u> Consider now general (i.e., not necessarily commuting)

$X_1, \dots, X_n \in (\mathcal{A}, \varphi)$, where

$\mathcal{A}$ = unital algebra

$\varphi : \mathcal{A} \to \mathbb{C}$ unital linear functional

(usually with some more analytic structure)

Can we give sense to $\mu_{X_1,\dots,X_n}$ in this setting.

The only item which makes directly sense is the combinatorial item iv), and this serves as our definition in the non-commutative case:

$\mu_{X_1,\dots,X_n}$:= the collection of all moments $\varphi(X_{i(1)} \cdots X_{i(k)})$ $\left(\substack{k \in \mathbb{N} \\ 1 \le i(1), \dots, i(k) \le n} \right)$ of our (non-commutative) random variables $X_1, \dots, X_n$

Our goal is an analytic understanding of this, i.e. to find non-commutative versions/replacements for items (i)–(iii)

In particular, we would like to have notions for and results on

- "smoothness" or "regularity" of nc distributions
- absence of "atoms"
- existence of "densities"

We still don't know what a

"non-commutative probability measure"

is, but there was quite some progress on dealing with this via nc versions of (ii) and (iii) in recent years. In particular, we can say quite a bit about the distribution of

$f(X_1, \dots, X_n)$ for big classes of $(X_1, \dots, X_n)$ and big classes of f.

In particular, this gives results on the asymptotic eigenvalue distribution for polynomials in independent random matrices (or, equivalently, the distribution of polynomials in free variables)

These results rely in particular on progress on

- operator-valued version of free probability theory of Voiculescu
- free analysis or free nc function theory

  ↑ Voiculescu — ↑ Kaliuzhnyi-Verbovetskyi + Vinnikov

- relating analytic questions about operators in vN-algebras with theory (of Cohn et al) of non-commutative linear algebra and free skew field

Much of this was achieved in recent years in the context of ERC-grant on "Non-Commutative Distributions" (2014-19)