CHAPTER V

OPERATOR-VALUED STOCHASTIC PROCESSES AND STOCHASTIC DIFFERENTIAL EQUATIONS

In this chapter, we will focus on another philosophy behind the concept of cumulants. Namely, even if there is no free product or free convolution around, it may in some problems be quite useful to make a transition from moments to cumulants. The reason for this lies in the fact that some properties of our distributions have a more handable formulation in terms of cumulants than in terms of moments. The most important of such properties is the product property which translates for the cumulants into a cluster property. The frame where such transitions from moments to cumulants become important is the theory of stochastic differential equations. Up to now, two other forms of cumulants, the 'ordered' and the 'partial' ones, have been used in such a context. We will compare these cumulants with the ones which we have introduced in our investigations and which will from now on be called 'non-crossing cumulants'. We will point out that in the operator-valued case the non-crossing cumulants are the only ones which behave nicely in all respects.

5.1. B-valued stochastic processes

To set our frame, we have to introduce, as a generalization of the concepts of the last chapter, the notion of a B-valued stochastic process and its distribution. Again, cumulants will play a prominent role.

5.1.1. Definition. 1) Let (A, φ) be a probability space over B. A family $\{a(t) \mid t \in \mathbb{R}\}$ of B-valued random variables will be called a *B-valued stochastic process*.
2) Let

$$B\langle \mathcal{X}_{\mathbb{R}} \rangle := *_{\substack{B \\ t\in\mathbb{R}}} B\langle X(t) \rangle$$

be the algebra generated freely by B and infinitely many non-commuting indeterminates $X(t)$ $(t \in \mathbb{R})$. For a B-valued stochastic process $\{a(t) \mid t \in \mathbb{R}\} \subset (A, \varphi)$ we define the B-functional

$$\nu_{\{a(t)\}} : B\langle \mathcal{X}_{\mathbb{R}} \rangle \to B \qquad \text{by} \qquad \nu_{\{a(t)\}} = \varphi \circ \tau_{\{a(t)\}},$$

where $\tau_{\{a(t)\}} : B\langle \mathcal{X}_{\mathbb{R}} \rangle \to A$ is the unique homomorphism such that $\tau_{\{a(t)\}}(b) = b$ for all $b \in B$ and $\tau_{\{a(t)\}}(X(t)) = a(t)$ for all $t \in \mathbb{R}$. This $\nu_{\{a(t)\}}$ is called the *distribution* of the B-valued stochastic process $\{a(t) \mid t \in \mathbb{R}\}$. The set of all possible distributions of B-valued stochastic processes will be denoted by $\Sigma_B^{(\mathbb{R})}$,

$$\Sigma_B^{(\mathbb{R})} := \{\nu : B\langle \mathcal{X}_{\mathbb{R}} \rangle \to B \mid \nu \ B\text{-functional}\}.$$

As before, we will mainly be interested in the case of positive distributions.

5.1.2. NOTATION. If B is a C^*-algebra, then we equip $B\langle\mathcal{X}_\mathbb{R}\rangle$ with the canonical $*$-structure given by $X(t)^* = X(t)$ for all $t \in \mathbb{R}$ and denote the set of positive distributions of B-valued stochastic processes by $\Sigma_B^{(\mathbb{R})+}$,

$$\Sigma_B^{(\mathbb{R})+} := \{\nu \in \Sigma_B^{(\mathbb{R})} \mid \nu \text{ positive}\}.$$

One could now go on as in chapter 4 and define the free convolution of two B-valued stochastic processes and repeat for this situation more or less the same analysis as for B-valued random variables. But this brings nothing new, one only has to replace $\mathcal{X}_m$ by $\mathcal{X}_\mathbb{R}$.

Instead of this, we will now focus on the above mentioned idea that, even if there is no freeness or free convolution around, the transition from a process to its cumulants may be quite useful in some cases.

5.1.3. NOTATIONS. 1) Again, all information on the distribution $\nu = \nu_{\{a(t)\}}$ of a B-valued stochastic process $\{a(t) \mid t \in \mathbb{R}\} \subset (A, \varphi)$ is contained in its *cumulants*

$$\xi(\nu) := \big(\xi^{(\nu)}_{n;t_0,\ldots,t_n}\big)_{n\in\mathbb{N};t_0,\ldots,t_n\in\mathbb{R}},$$

where, for all $n \in \mathbb{N}_0$ and $t_0, \ldots, t_n \in \mathbb{R}$,

$$\xi^{(\nu)}_{n;t_0,\ldots,t_n} : \underbrace{B \times \cdots \times B}_{n\text{-times}} \to B$$

are linear mappings given by $(b_1, \ldots, b_n \in B)$

$$\begin{aligned}\xi^{(\nu)}_{n;t_0,\ldots,t_n}(b_1, \ldots, b_n) :&= c_\nu^{(n+1)}\big(X(t_0) \otimes b_1X(t_1) \otimes \cdots \otimes b_nX(t_n)\big)\\ &= c^{(n+1)}\big(a(t_0) \otimes b_1a(t_1) \otimes \cdots \otimes b_na(t_n)\big),\end{aligned}$$

where $\hat{c} = (c^{(n)}) \in \mathbf{I}^c(A, B)$ is the cumulant function of φ. In particular,

$$\xi^{(\nu)}_{0;t} : \mathbb{C} \to B, \qquad \xi^{(\nu)}_{0;t}(1) = c_\nu^{(1)}\big(X(t)\big) = \nu\big(X(t)\big) = \varphi\big(a(t)\big).$$

2) Let us denote by $B\langle\mathcal{X}_\mathbb{R}\rangle_0$ the B-B-bimodule (we use the notation $\mathcal{X}_\mathbb{R} := \operatorname{span}\{X_t \mid t \in \mathbb{R}\} = \{\sum_{\text{finite}} \alpha_i X(t_i) \mid t_i \in \mathbb{R}, \alpha_i \in \mathbb{C}\}$)

$$B\langle\mathcal{X}_\mathbb{R}\rangle_0 := B\mathcal{X}_\mathbb{R}B \oplus B\mathcal{X}_\mathbb{R}B\mathcal{X}_\mathbb{R}B \oplus B\mathcal{X}_\mathbb{R}B\mathcal{X}_\mathbb{R}B\mathcal{X}_\mathbb{R}B \oplus \cdots \subset B\langle\mathcal{X}_\mathbb{R}\rangle,$$

i.e. those polynomials in $B\langle\mathcal{X}_\mathbb{R}\rangle$ without constant term. Then we can consider $\xi(\nu)$ also as a bimodule map

$$\xi(\nu) : B\langle\mathcal{X}_\mathbb{R}\rangle_0 \to B$$

with $(n \in \mathbb{N},\ t_0, \ldots, t_n \in \mathbb{R},\ b_0, \ldots, b_{n+1} \in B)$

$$\begin{aligned}\xi(\nu)\big(b_0X(t_0)b_1X(t_1)\ldots b_nX(t_n)b_{n+1}\big) &= b_0\big(\xi^{(\nu)}_{n;t_0,\ldots,t_n}(b_1, \ldots, b_n)\big)b_{n+1}\\ &= c_\nu^{(n+1)}\big(b_0X(t_0) \otimes b_1X(t_1) \otimes \cdots \otimes b_nX(t_n)b_{n+1}\big)\\ &= c^{(n+1)}\big(b_0a(t_0) \otimes b_1a(t_1) \otimes \cdots \otimes b_na(t_n)b_{n+1}\big).\end{aligned}$$

5.1.4. Definition. 1) Let $\{\hat{a}(t) \mid t \in \mathbb{R}\}$ be a B-valued stochastic process. If the cumulants $\xi := \xi(\nu_{\{\hat{a}(t)\}})$ of $\{\hat{a}(t) \mid t \in \mathbb{R}\}$ have the property

$$\xi_{n;t_0,\ldots,t_n} \equiv 0 \qquad \text{for } n \neq 1 \text{ and all } t_0, \ldots, t_n \in \mathbb{R}$$

and if

$$\xi_{1;s,t}(b) = \nu_{\{\hat{a}(t)\}}\big(X(s)bX(t)\big) = \eta_{st}(b)$$

for all $s, t \in \mathbb{R}$ and $b \in B$, with linear mappings (for all $s, t \in \mathbb{R}$)

$$\eta_{st} : B \to B,$$

then we call $\{\hat{a}(t) \mid t \in \mathbb{R}\}$ a centered *B-Gaussian stochastic process* with covariance $\eta := (\eta_{st})_{s,t\in\mathbb{R}}$.
2) Let $\{a(t) \mid t \in \mathbb{R}\}$ be a B-valued stochastic process with covariance ($s, t \in \mathbb{R}$, $b \in B$)

$$\eta_{st}(b) := \nu_{\{a(t)\}}\big(X(s)bX(t)\big).$$

If $\{\hat{a}(t) \mid t \in \mathbb{R}\}$ is a B-Gaussian stochastic process with covariance $\eta = (\eta_{st})_{s,t\in\mathbb{R}}$, then we call $\{\hat{a}(t) \mid t \in \mathbb{R}\}$ a *Gaussian approximation* of $\{a(t) \mid t \in \mathbb{R}\}$.

It is clear that for all covariance matrices there exist corresponding B-Gaussian stochastic processes, i.e. for all B-valued stochastic processes one can construct Gaussian approximations. But what is more important, we have the analogue of Theorem 4.3.1 showing that positivity is preserved in making the Gaussian approximation (and thus no positivity problem [HS] as in the corresponding approximation for ordered cumulants appears, see section 5.4).

5.1.5. Theorem. *Let $\{a(t) \mid t \in \mathbb{R}\}$ be a B-valued stochastic process and $\{\hat{a}(t) \mid t \in \mathbb{R}\}$ a corresponding Gaussian approximation. If the distribution of $\{a(t) \mid t \in \mathbb{R}\}$ is positive, then the distribution of $\{\hat{a}(t) \mid t \in \mathbb{R}\}$ is positive, too:*

$$\nu_{\{a(t)\}} \in \Sigma_B^{(\mathbb{R})+} \Longrightarrow \nu_{\{\hat{a}(t)\}} \in \Sigma_B^{(\mathbb{R})+}.$$

Proof. This can be shown like in the proof of Theorem 4.3.1 by central limit arguments and an arbitrary positive distribution having the right covariance or one can construct directly on a full Fock space the Gaussian approximation. Namely, let $\eta = (\eta_{st})_{s,t\in\mathbb{R}}$ be the covariance of $\{a(t) \mid t \in \mathbb{R}\}$. Then we define on the full Fock space

$$\mathcal{F}(B\mathcal{X}_{\mathbb{R}}B) := B\langle\mathcal{X}_{\mathbb{R}}\rangle = B \oplus B\mathcal{X}_{\mathbb{R}}B \oplus B\mathcal{X}_{\mathbb{R}}B\mathcal{X}_{\mathbb{R}}B \oplus \ldots$$

a B-valued inner product

$$\langle\ ,\ \rangle : B\langle\mathcal{X}_{\mathbb{R}}\rangle \times B\langle\mathcal{X}_{\mathbb{R}}\rangle \to B$$

by linear extension of

$$\begin{aligned}\big\langle b_0X(s_0)b_1X(s_1)\ldots b_nX(s_n)b_{n+1}, \tilde{b}_0X(t_0)\tilde{b}_1X(t_1)\ldots\tilde{b}_kX(t_k)b_{k+1}\big\rangle :=\\ = \delta_{nk}b_{n+1}^*\eta_{s_nt_n}\Big(b_n^*\ldots\eta_{s_1t_1}\big(b_1^*\eta_{s_0t_0}(b_0^*\tilde{b}_0)\tilde{b}_1\big)\ldots\tilde{b}_n\Big)\tilde{b}_{n+1}.\end{aligned}$$

The given η fulfills the analogue of the condition in Theorem 4.3.1 implying the positivity of our inner product and thus the positivity of the vacuum expectation φ on the $*$-algebra A generated by B and all creation and annihilation operators. The latter are defined as usual, i.e.

$$l^*(t)b_0X(t_0)\dots X(t_n)b_{n+1} = X(t)b_0X(t_0)\dots X(t_n)b_{n+1}$$

and

$$l(t)b = 0$$
$$l(t)b_0X(t_0)b_1X(t_1)\dots X(t_n)b_{n+1} = \big(\eta_{tt_0}(b_0)b_1\big)X(t_1)\dots X(t_n)b_{n+1}.$$

As in the proof of Theorem 4.7.1, one can check that the process

$$\{l(t) + l^*(t) \mid t \in \mathbb{R}\} \subset (A, \varphi)$$

gives the wanted B-Gaussian stochastic process with covariance η. $\square$

5.2. Formulation of the problem

In physical contextes, one often encounters the following problem: For a given B-valued stochastic process

$$\{L(t) \mid t \in \mathbb{R}\} \subset (A, \varphi)$$

one considers a multiplicative stochastic differential equation for a wanted B-valued stochastic process $\{U(t) \mid t \in \mathbb{R}\}$ of the form

$$\frac{d}{dt}U(t) = L(t)U(t) \quad (t \geq 0), \qquad U(s) = 1 \in B \subset A \quad (s \leq 0).$$

Usually, one is not so much interested in the process $\{U(t) \mid t \in \mathbb{R}\}$ itself, but one would be satisfied with having a good knowledge on some average behaviour of $U(t)$, in particular one wants to know the time development of the mean $\langle U(t)\rangle := \varphi(U(t))$, see, e.g., [HS,Kub,NSp1,NSp2,ShA,Ter,vKa1,vKa2].

The formal solution of our stochastic differential equation is given by

$$U(t) = \sum_{n=0}^{\infty} \int\!\!\dots\!\!\int_{t\geq t_1\geq\dots\geq t_n\geq 0} L(t_1)\dots L(t_n)dt_1\dots dt_n,$$

yielding

$$\langle U(t)\rangle = \sum_{n=0}^{\infty} \int\!\!\dots\!\!\int_{t\geq t_1\geq\dots\geq t_n\geq 0} \varphi\big(L(t_1)\dots L(t_n)\big)dt_1\dots dt_n.$$

We do not bother about the question of convergence of this series, but just assume that all this makes sense (which, of course, has to be checked in a concrete problem).

The problem with the above expansion is now that it is usually quite useless. It should be clear that it can be calculated directly only in the simplest cases. For going beyond such simple (and usual uninteresting) examples one has to make

approximations, but the crucial deficiency of the above series is that it is inadequate for approximation techniques.

In most physical applications one can assume a product property for the potential $L(t)$, saying that $\langle L(t_1)\dots L(t_n)\rangle$ factorizes into $\langle L(t_1)\dots L(t_k)\rangle$ $\langle L(t_{k+1})\dots L(t_n)\rangle$ if the variables $t_1,\dots,t_n$ are grouped into two clusters $t_1,\dots,t_k$ and $t_{k+1},\dots,t_n$ with a sufficently large separation between these two clusters. This may, for example, be fulfilled, if we have $|t_i - t_j| > \tau_c$ for all $i = 1,\dots,k$ and $j = k+1,\dots,n$, where τ_c is the auto-correlation time of the potential.

On the level of our above series this information does not help anything since factorization does not lead to smallness of terms in the integrand. But since, by Prop. 2.5.3, the product property of the moments translates into the cluster property for the cumulants, it seems plausible that, instead of an expansion of $\langle U(t)\rangle$ in terms of moments, an expansion in terms of cumulants provides the proper truncation criterium for approximation techniques.

5.3. Possible solutions of the problem

We have to face the following problem: For a distribution $\nu \in \Sigma_B^{(\mathbb{R})}$ of a B-valued stochastic process we want to rewrite the formal series ($t, s \in \mathbb{R}$, $t \geq s$)

$$\langle U(t,s)\rangle := \sum_{n=0}^{\infty} \int\!\cdots\!\int_{t\geq t_1\geq\cdots\geq t_n\geq s} \nu\big(X(t_1)\dots X(t_n)\big)\,dt_1\dots dt_n \in B$$

in a form which is more suitable for approximations. As we argued in the last section, an expansion in cumulants is a good candidate for such a reformulation. Apart from the cumulants which we have used till now and which will from now on be called non-crossing cumulants, there exist two other forms of cumulants which are commonly used in this context. These are the so-called ordered cumulants and partial cumulants. We will now compare the corresponding three possibilities for rewriting the above series.

In the same way as the non-crossing cumulants are determined by the lattice of non-crossing partitions, the ordered and partial cumulants are determined by the lattice of all partitions and the lattice of interval partitions, respectively. We will not focus on this lattice point of view but instead we use a recursion formula containing the same information for the definition of the respective cumulants. For our non-crossing cumulants this is nothing else but Theorem 2.2.2.

5.3.1. Definition. Let $\nu \in \Sigma_B^{(\mathbb{R})}$ be a distribution of a B-valued stochastic process.
1) The *ordered cumulants* of ν

$$\xi^{\mathrm{ord}} : \mathbb{C}\langle \mathcal{X}_{\mathbb{R}}\rangle_0 \to B$$

are defined by the recurrence relation ($n \in \mathbb{N}$, $t, t_1, \dots, t_n \in \mathbb{R}$)

$$\nu\big(X(t)X(t_1)\dots X(t_n)\big) =$$
$$= \sum_{r=0}^{n} \sum_{\substack{(i(1),\dots,i(r))\\ \cup(j(1),\dots,j(n-r))\\ =(1,\dots,n)}} \xi^{\mathrm{ord}}\big(X(t)X(t_{i(1)})\dots X(t_{i(r)})\big)\cdot \nu\big(X(t_{j(1)})\dots X(t_{j(n-r)})\big),$$

where we have to sum over all decompositions of $(1,\dots,n)$ into two ordered subsets $\big(i(1),\dots,i(r)\big)$ and $\big(j(1),\dots,j(n-r)\big)$, each of which may be empty. In particular we have

$$\xi^{\mathrm{ord}}\big(X(t)\big) = \nu\big(X(t)\big) \qquad \text{for all } t \in \mathbb{R}.$$

2) The *partial cumulants* of ν

$$\xi^{\mathrm{par}} : B\langle \mathcal{X}_{\mathbb{R}} \rangle_0 \to B$$

are defined by the recurrence relation ($n \in \mathbb{N}$, $t, t_1, \dots, t_n \in \mathbb{R}$, $b_0, \dots, b_{n+1} \in B$)

$$\begin{aligned}
&\nu\big(b_0 X(t) b_1 X(t_1) b_2 \dots b_n X(t_n) b_{n+1}\big) = \\
&= \sum_{r=0}^{n} \xi^{\mathrm{par}}\big(b_0 X(t) b_1 X(t_1) b_2 \dots b_r X(t_r)\big) \cdot \nu\big(b_{r+1} X(t_{r+1}) b_{r+2} \dots b_n X(t_n) b_{n+1}\big).
\end{aligned}$$

In particular we have

$$\xi^{\mathrm{par}}\big(b_0 X(t) b_1\big) = \nu\big(b_0 X(t) b_1\big) = b_0 \nu\big(X(t)\big) b_1 \qquad \text{for all } t \in \mathbb{R} \text{ and } b_0, b_1 \in B.$$

3) The *non-crossing cumulants* of ν

$$\xi^{\mathrm{NC}} : B\langle \mathcal{X}_{\mathbb{R}} \rangle_0 \to B$$

are defined by the recurrence relation ($n \in \mathbb{N}$, $t, t_1, \dots, t_n \in \mathbb{R}$, $b_0, \dots, b_{n+1} \in B$)

$$\begin{aligned}
&\nu\big(b_0 X(t) b_1 X(t_1) b_2 \dots b_n X(t_n) b_{n+1}\big) = \\
&= \sum_{r=0}^{n} \sum_{\substack{(i(1),\dots,i(r)) \\ \subset (1,\dots,n)}} \xi^{\mathrm{NC}}\Big(b_0 X(t) b_1 \cdot \nu\big(X(t_1) b_2 \dots X(t_{i(1)-1})\big) \\
&\qquad b_{i(1)} X(t_{i(1)}) b_{i(1)+1} \cdot \nu\big(X(t_{i(1)+1}) b_{i(1)+2} \dots X(t_{i(2)-1})\big) \\
&\qquad\qquad b_{i(2)} X(t_{i(2)}) \dots \nu\big(X(t_{i(r-1)+1}) b_{i(r-1)+2} \dots X(t_{i(r)-1})\big) \\
&\qquad\qquad b_{i(r)} X(t_{i(r)}) b_{i(r)+1} \cdot \nu\big(X(t_{i(r)+1}) b_{i(r)+2} \dots X(t_n)\big) b_{n+1}\Big).
\end{aligned}$$

In particular, we have

$$\xi^{\mathrm{NC}}\big(b_0 X(t) b_1\big) = \nu\big(b_0 X(t) b_1\big) = b_0 \nu\big(X(t)\big) b_1 \qquad \text{for all } t \in \mathbb{R} \text{ and } b_0, b_1 \in B.$$

5.3.2. Examples. 1) In the two lowest orders, all three types of cumulants are the same, namely

$$\xi^{\mathrm{ord}}\big(X(t)\big) = \xi^{\mathrm{par}}\big(X(t)\big) = \xi^{\mathrm{NC}}\big(X(t)\big) = \nu\big(X(t)\big)$$

and

$$\begin{aligned}
\xi^{\mathrm{ord}}\big(X(t_1)X(t_2)\big) = \xi^{\mathrm{par}}\big(X(t_1)X(t_2)\big) &= \xi^{\mathrm{NC}}\big(X(t_1)X(t_2)\big) \\
&= \nu\big(X(t_1)X(t_2)\big) - \nu\big(X(t_1)\big) \cdot \nu\big(X(t_2)\big).
\end{aligned}$$

2) The cumulants of third order have the form

$$\begin{aligned}\xi^{\mathrm{ord}}\big(X(t_1)X(t_2)X(t_3)\big) &= \nu\big(X(t_1)X(t_2)X(t_3)\big)\\ &\quad - \nu\big(X(t_1)X(t_2)\big)\cdot\nu\big(X(t_3)\big)\\ &\quad - \nu\big(X(t_1)\big)\cdot\nu\big(X(t_2)X(t_3)\big)\\ &\quad - \nu\big(X(t_1)X(t_3)\big)\cdot\nu\big(X(t_2)\big)\\ &\quad + \nu\big(X(t_1)\big)\cdot\nu\big(X(t_2)\big)\cdot\nu\big(X(t_3)\big)\\ &\quad + \nu\big(X(t_1)\big)\cdot\nu\big(X(t_3)\big)\cdot\nu\big(X(t_2)\big),\end{aligned}$$

$$\begin{aligned}\xi^{\mathrm{par}}\big(X(t_1)X(t_2)X(t_3)\big) &= \nu\big(X(t_1)X(t_2)X(t_3)\big)\\ &\quad - \nu\big(X(t_1)X(t_2)\big)\cdot\nu\big(X(t_3)\big)\\ &\quad - \nu\big(X(t_1)\big)\cdot\nu\big(X(t_2)X(t_3)\big)\\ &\quad + \nu\big(X(t_1)\big)\cdot\nu\big(X(t_2)\big)\cdot\nu\big(X(t_3)\big),\end{aligned}$$

$$\begin{aligned}\xi^{\mathrm{NC}}\big(X(t_1)X(t_2)X(t_3)\big) &= \nu\big(X(t_1)X(t_2)X(t_3)\big)\\ &\quad - \nu\big(X(t_1)X(t_2)\big)\cdot\nu\big(X(t_3)\big)\\ &\quad - \nu\big(X(t_1)\big)\cdot\nu\big(X(t_2)X(t_3)\big)\\ &\quad - \nu\Big(X(t_1)\nu\big(X(t_2)\big)X(t_3)\Big)\\ &\quad + 2\nu\big(X(t_1)\big)\cdot\nu\big(X(t_2)\big)\cdot\nu\big(X(t_3)\big),\end{aligned}$$

3) The cumulants of order four give in the special case of a centered process, i.e. $\nu\big(X(t)\big) = 0$ for all $t \in \mathbb{R}$, the following expressions.

$$\begin{aligned}\xi^{\mathrm{ord}}\big(X(t_1)X(t_2)X(t_3)X(t_4)\big) &= \nu\big(X(t_1)X(t_2)X(t_3)X(t_4)\big)\\ &\quad - \nu\big(X(t_1)X(t_2)\big)\cdot\nu\big(X(t_3)X(t_4)\big)\\ &\quad - \nu\big(X(t_1)X(t_3)\big)\cdot\nu\big(X(t_2)X(t_4)\big)\\ &\quad - \nu\big(X(t_1)X(t_4)\big)\cdot\nu\big(X(t_2)X(t_3)\big)\\ \xi^{\mathrm{par}}\big(X(t_1)X(t_2)X(t_3)X(t_4)\big) &= \nu\big(X(t_1)X(t_2)X(t_3)X(t_4)\big)\\ &\quad - \nu\big(X(t_1)X(t_2)\big)\cdot\nu\big(X(t_3)X(t_4)\big)\\ \xi^{\mathrm{NC}}\big(X(t_1)X(t_2)X(t_3)X(t_4)\big) &= \nu\big(X(t_1)X(t_2)X(t_3)X(t_4)\big)\\ &\quad - \nu\big(X(t_1)X(t_2)\big)\cdot\nu\big(X(t_3)X(t_4)\big)\\ &\quad - \nu\Big(X(t_1)\nu\big(X(t_2)X(t_3)\big)X(t_4)\Big).\end{aligned}$$

5.3.3. Remarks. 1) Since in our formal series for $\langle U(t,s)\rangle$ only expressions like $\nu\big(X(t_1)X(t_2)\dots X(t_n)\big)$ but no terms like $\nu\big(b_0X(t_1)b_1X(t_2)\dots X(t_n)b_n\big)$ (for $b_0,\dots,b_n \in B$) appear, it suffices for our problem to define the cumulants as linear maps on $\mathbb{C}\langle\mathcal{X}_{\mathbb{R}}\rangle_0$. But as we see from the structure of the recurrence formula, at least for the non-crossing cumulants we are forced in any case to extend the definition as a bimodule map to $B\langle\mathcal{X}_{\mathbb{R}}\rangle_0$. For the partial cumulants, such an extension is

not necessary, but it is clear that it presents no problems. The ordered cumulants, however, do not allow this generalization, because in contrast to the case of the partial and the non-crossing cumulants, the time-ordering of the arguments is destroyed in the definition of the ordered cumulants. As the name 'ordered' indicates one has to change the given order and introduce an artificial one by separating the moments and the cumulants in the recurrence formula. This can be seen clearly from the foregoing examples and is the reason for the positivity problem encountered in the corresponding Gaussian approximation, see the next section.

2) The main feature of the three types of cumulants which makes them useful in the context of our problem is the cluster property. For the non-crossing cumulants we have proved this property in Prop. 2.5.3, but a similar statement is also true for the ordered and partial cumulants, see [vKa1,Ter,HS]. The proof of the cluster property requires essentially the fact that cumulants and moments are connected by some recurrence formula, the concrete form of this formula is not so important.

3) The ordered cumulants reduce in the scalar-valued case to the usual cumulants, which are widely used in classical probability theory. The generalization to the operator-valued case is due to Kubo [Kub] and van Kampen [vKa1] and was introduced in order to deal with the problem considered here. The partial cumulants were also introduced in such a context, namely by Terwiel [Ter] and by von Waldenfels [vWa1,vWa2].

4) In the same way as the free convolution is linearized by the non-crossing cumulants the usual convolution is linearized by the ordered cumulants. By analogy, one may wonder whether there exists also a form of convolution which is linearized by the partial cumulants. Such a 'boolean' convolution was just recently developed in [Wor,SpW]. As the free convolution and the usual convolution are connected with the free product and the tensor product, respectively, this boolean convolution is derived from a 'boolean' product, which can be traced back to investigations of Bożejko [Boz1,Boz2], see also [BSp,BLS].

5) We know by Remark 3.3.2 that elements $b \in B$ are free with respect to everything. On the level of our non-crossing cumulants this translates into the statement that the cumulants $\xi^{\mathrm{NC}}_{\{a(t)\}}$ of an arbitrary process $\{a(t) \mid t \in \mathbb{R}\} \subset (A, \varphi)$ and the cumulants $\xi^{\mathrm{NC}}_{\{\hat{a}(t)\}}$ of a shifted process $\{\hat{a}(t) := a(t) + b(t) \mid t \in \mathbb{R}\} \subset (A, \varphi)$ with some $b(t) \in B$ $(t \in \mathbb{R})$ differ only in the first order, i.e.

$$\xi^{\mathrm{NC}}_{\{\hat{a}(t)\}}\big(X(t_0)\big) = \xi^{\mathrm{NC}}_{\{a(t)\}}\big(X(t_0)\big) + b(t_0)$$

for all $t_0 \in \mathbb{R}$ and

$$\xi^{\mathrm{NC}}_{\{\hat{a}(t)\}}\big(X(t_0)b_1 \ldots b_n X(t_n)\big) = \xi^{\mathrm{NC}}_{\{a(t)\}}\big(X(t_0)b_1 \ldots b_n X(t_n)\big)$$

for all $n \geq 1$, $t_0, \ldots, t_n \in \mathbb{R}$, and $b_1, \ldots, b_n \in B$. For the ordered and partial cumulants, however, this property is not fulfilled. Let us check this for the cumulants of order three under the assumption that the process $\{a(t) \mid t \in \mathbb{R}\}$ is symmetric, i.e. that all odd moments vanish. Then we calculate with the help of Example 5.3.2 that

$$\xi^{\mathrm{ord}}_{\{\hat{a}(t)\}}\big(X(t_1)X(t_2)X(t_3)\big) = \varphi\big(a(t_1)b(t_2)a(t_3)\big) - \varphi\big(a(t_1)a(t_3)b(t_2)\big)$$

and

$$\xi^{\mathrm{par}}_{\{\hat{a}(t)\}}\big(X(t_1)X(t_2)X(t_3)\big) = \varphi\big(a(t_1)b(t_2)a(t_3)\big).$$

Only

$$\xi^{\mathrm{NC}}_{\{\hat{a}(t)\}}\big(X(t_1)X(t_2)X(t_3)\big) = 0$$

gives the 'right' answer, thus a shift with elements from B cannot be described easily in terms of ordered or partial cumulants. One sees from the above example that for the ordered cumulants this deficiency comes again from the destruction of the time ordering and thus it is a characteristicum of the non-commutativity of the general operator-valued case. If we restrict to the scalar-valued case $B = \mathbb{C}$, then everything works fine, as it has to, since we know from classical probability theory that the ordinary convolution with a delta distribution δ_c gives nothing else than a shift by the amount c. For the partial cumulants, however, the deficiency does not come from the non-commutativity of our process with elements in B but from the structure of the recursion formula or, to put it in another way, from the structure of the lattice of interval partitions. Thus in this case even the scalar-valued boolean convolution shows the strange feature that convolution with a delta distribution is not merely a shift, but a quite complicated operation.

5.3.4. Theorem. *Let $\nu \in \Sigma_B^{(\mathbb{R})}$ be a distribution of a B-valued stochastic process and define a formal series by ($t, s \in \mathbb{R}$, $t \geq s$)*

$$\langle U(t,s)\rangle := \sum_{n=0}^{\infty} \int\limits_{t\geq t_1\geq\cdots\geq t_n\geq s} \cdots \int \nu\big(X(t_1)\ldots X(t_n)\big)dt_1\ldots dt_n \in B.$$

Then, with

$$\xi^{\mathrm{ord}} := \xi^{\mathrm{ord}}(\nu), \qquad \xi^{\mathrm{par}} := \xi^{\mathrm{par}}(\nu), \qquad \xi^{\mathrm{NC}} := \xi^{\mathrm{NC}}(\nu)$$

denoting the ordered, partial, and non-crossing cumulants, respectively, the above definition of $\langle U(t,s)\rangle$ is equivalent to each of the following three (integro-) differential equations (understood in a formal sense)

$$\frac{d}{dt}\langle U(t,s)\rangle = \Big\{\sum_{n=0}^{\infty} \int\limits_{t\geq t_1\geq\cdots\geq t_n\geq s} \cdots \int \xi^{\mathrm{ord}}\big(X(t)X(t_1)\ldots X(t_n)\big)dt_1\ldots dt_n\Big\}\langle U(t,s)\rangle$$

$$\frac{d}{dt}\langle U(t,s)\rangle = \sum_{n=0}^{\infty} \int\limits_{t\geq t_1\geq\cdots\geq t_n\geq s} \cdots \int \xi^{\mathrm{par}}\big(X(t)X(t_1)\ldots X(t_n)\big)dt_1\ldots dt_{n-1}\langle U(t_n,s)\rangle dt_n$$

$$\frac{d}{dt}\langle U(t,s)\rangle = \sum_{n=0}^{\infty} \int\limits_{t\geq t_1\geq\cdots\geq t_n\geq s} \cdots \int \xi^{\mathrm{NC}}\Big(X(t)\langle U(t,t_1)\rangle X(t_1)\langle U(t_1,t_2)\rangle\ldots$$
$$\ldots\langle U(t_{n-1},t_n)\rangle X(t_n)\langle U(t_n,s)\rangle\Big)dt_1\ldots dt_n.$$

Proof. This follows directly by inserting the recursion formulas and by regrouping terms and changing integration variables. □

5.3.5. Remark. The equations corresponding to the ordered and partial cumulants are due to van Kampen [vKa1] and Terwiel [Ter] (compare also [vWa1]), respectively. For a comparison of these two cases see also [ShA,HS]. The expansion formula in terms of the non-crossing cumulants in this general operator-valued case is new, the special scalar-valued case was treated in [NSp1,NSp2].

5.4. Gaussian approximation

The most prominent approximation (and usually the only one which is technically handable) for our expansion formulas is the so-called Gaussian approximation, where one only retains the cumulants of first and second order. In this case our expansion formulas reduce to

$$\begin{aligned}
\frac{d}{dt}\langle U(t,s)\rangle &= \nu\big(X(t)\big)\langle U(t,s)\rangle + \Big\{\int_s^t \xi\big(X(t)X(t_1)\big)dt_1\Big\}\langle U(t,s)\rangle \\
\frac{d}{dt}\langle U(t,s)\rangle &= \nu\big(X(t)\big)\langle U(t,s)\rangle + \int_s^t \xi\big(X(t)X(t_1)\big)\langle U(t_1,s)\rangle dt_1 \\
\frac{d}{dt}\langle U(t,s)\rangle &= \nu\big(X(t)\big)\langle U(t,s)\rangle + \int_s^t \xi\big(X(t)\langle U(t,t_1)\rangle X(t_1)\big)\langle U(t_1,s)\rangle dt_1,
\end{aligned}$$

where

$$\xi\big(X(t)bX(t_1)\big) := \nu\big(X(t)bX(t_1)\big) - \nu\big(X(t)\big)b\nu\big(X(t_1)\big)$$

is the common value of all three cumulants of second order. For the application of such formulas in physical contextes we refer to [CSh,Chv,ESh,NSp1,ShA,vKa2].

The Gaussian approximation in the case of the non-crossing cumulants reduces our expansion formula to the so-called Kraichnan equation [Kra,FB,NSp1,NSp2], which appears usually by some obscure ad hoc approximations. Thus our concept of non-crossing cumulants and the corresponding expansion formula give a well defined frame for a better conceptual understanding of the Kraichnan equation.

The Gaussian approximation means that we replace our process by another process which has the Gaussian property with respect to the respective cumulants, i.e. for which the respective cumulants of order higher than two vanish. For the non-crossing cumulants this is exactly (modulo centering) what we called Gaussian approximation in Def. 5.1.4. An important point is the question whether such an approximation preserves positivity. Since physical distributions are always positive, this amounts to the question whether we replace the original problem by some other physical problem or only by some mathematical truncation. In the latter case the solution of the approximated expansion formula might show some unphysical features.

For the non-crossing cumulants our Theorem 5.1.5 assures that the Gaussian approximation of a positive distribution is indeed positive and thus physically meaningful. The same is true for the Gaussian approximation in the case of the partial cumulants. For the ordered cumulants, however, we may loose positivity in making such an approximation as pointed out by Hegerfeldt and Schulze [HS]. This is intimately connected with the destruction of the time order in the definition of the ordered cumulants. To see this more clearly, let us denote by ν^{ord}, ν^{par}, and ν^{NC} the Gaussian approximations for a distribution ν for the ordered, partial, and non-crossing cumulants. Then, for a centered distribution, Example 5.3.2 shows

that

$$\begin{aligned}
\nu^{\mathrm{ord}}\big(X(t_1)X(t_2)X(t_3)X(t_4)\big) &= \nu\big(X(t_1)X(t_2)\big)\cdot\nu\big(X(t_3)X(t_4)\big)\\
&\quad+\nu\big(X(t_1)X(t_3)\big)\cdot\nu\big(X(t_2)X(t_4)\big)\\
&\quad+\nu\big(X(t_1)X(t_4)\big)\cdot\nu\big(X(t_2)X(t_3)\big)\\
\nu^{\mathrm{par}}\big(X(t_1)X(t_2)X(t_3)X(t_4)\big) &= \nu\big(X(t_1)X(t_2)\big)\cdot\nu\big(X(t_3)X(t_4)\big)\\
\nu^{\mathrm{NC}}\big(X(t_1)X(t_2)X(t_3)X(t_4)\big) &= \nu\big(X(t_1)X(t_2)\big)\cdot\nu\big(X(t_3)X(t_4)\big)\\
&\quad+\nu\Big(X(t_1)\nu\big(X(t_2)X(t_3)\big)X(t_4)\Big).
\end{aligned}$$

If we now put $t_3 = t_2$ and $t_4 = t_1$, then, for ν positive, this yields

$$\nu^{\mathrm{par}}\big(X(t_1)X(t_2)X(t_2)X(t_1)\big) = \nu\big(X(t_1)X(t_2)\big)\cdot\nu\big(X(t_2)X(t_1)\big) \geq 0$$

and

$$\begin{aligned}
\nu^{\mathrm{NC}}\big(X(t_1)X(t_2)X(t_2)X(t_1)\big) &= \nu\big(X(t_1)X(t_2)\big)\cdot\nu\big(X(t_2)X(t_1)\big)\\
&\quad+\nu\Big(X(t_1)\nu\big(X(t_2)X(t_2)\big)X(t_1)\Big)\\
&\geq 0.
\end{aligned}$$

For ν^{ord}, however, we obtain

$$\begin{aligned}
\nu^{\mathrm{ord}}\big(X(t_1)X(t_2)X(t_2)X(t_1)\big) &= \nu\big(X(t_1)X(t_2)\big)\cdot\nu\big(X(t_2)X(t_1)\big)\\
&\quad+\nu\big(X(t_1)X(t_2)\big)\cdot\nu\big(X(t_2)X(t_1)\big)\\
&\quad+\nu\big(X(t_1)X(t_1)\big)\cdot\nu\big(X(t_2)X(t_2)\big),
\end{aligned}$$

which is in general, namely if $\nu\big(X(t_1)X(t_1)\big)$ and $\nu\big(X(t_2)X(t_2)\big)$ do not commute, not positive, thus showing that the 'ordered' Gaussian approximation does not preserve positivity in general.

These considerations together with Remarks 5.3.3 show that the non-crossing cumulants are the only ones which behave nicely in all respects and thus promise to be a natural and powerful tool for dealing with questions around generalized master equations.