CHAPTER II

OPERATOR-VALUED MULTIPLICATIVE FUNCTIONS ON THE LATTICE OF NON-CROSSING PARTITIONS

In this chapter, we present the combinatorial heart of our theory, namely the concept of operator-valued multiplicative functions. In the next chapter, two special types of such multiplicative functions will play the key role, namely the moment and the cumulant functions. Whereas these notions will require some additional multiplicative structure, we will deal here only with the linear structure and elaborate how much of the relation between moment and cumulant functions can be unterstood on this linear level.

2.1. Operator-valued multiplicative functions

In the following, B will always be a unital algebra and M a B-B-bimodule. By $M^{\otimes_B n}$ we denote the n-fold B-tensor product of M with itself, i.e.

$$M^{\otimes_B n} := \underbrace{M \otimes_B M \otimes_B M \otimes_B \cdots \otimes_B M}_{n\text{-times}} \qquad (n \in \mathbb{N}).$$

Note that $M^{\otimes_B n}$ is a B-B-bimodule, too. We also define $M^{\otimes_B 0} := B$.

2.1.1. Definition. Let, for each $n \in \mathbb{N}$,

$$f^{(n)} : M^{\otimes_B n} \to B$$

be a linear B-B-bimodule map. Then we define the corresponding (operator-valued) *multiplicative function*

$$\begin{aligned}\hat{f} = (f^{(n)})_{n\in\mathbb{N}} : \bigcup_{n=1}^{\infty} \big(NC(n) \times M^{\otimes_B n}\big) &\to B \\ (\pi, a_1 \otimes \cdots \otimes a_n) &\mapsto \hat{f}(\pi)[a_1 \otimes \cdots \otimes a_n]\end{aligned}$$

by the following recursive prescription: We can write $\pi \in NC(n)$ in the form $\pi = \pi_1 \cup \mathbf{1}_V$, where $V \in \pi$ is an interval in $S_{(n)}$, i.e. $V = [k,l]$ $(1 \le k \le l \le n)$. We put

$$\begin{aligned}&\hat{f}(\pi)[a_1 \otimes \cdots \otimes a_n] := \\ &\qquad = \hat{f}(\pi_1)\big[a_1 \otimes \cdots \otimes a_{k-1} f^{(l-k+1)}(a_k \otimes \cdots \otimes a_l) \otimes a_{l+1} \otimes \cdots \otimes a_n\big] \\ &\qquad = \hat{f}(\pi_1)\big[a_1 \otimes \cdots \otimes a_{k-1} \otimes f^{(l-k+1)}(a_k \otimes \cdots \otimes a_l) a_{l+1} \otimes \cdots \otimes a_n\big],\end{aligned}$$

where we set formally (for the case $\pi = \mathbf{1}_V$, $\pi_1 = \emptyset$)

$$\hat{f}(\emptyset)[b] = b \qquad (b \in B = M^{\otimes_B 0}).$$

Of course, $f^{(n)}$ is determined by $\hat{f}$ via

$$f^{(n)}(a_1 \otimes \cdots \otimes a_n) = \hat{f}(\mathbf{1}_n)[a_1 \otimes \cdots \otimes a_n].$$

Note that (for $\pi \in NC(n)$) our definition of $\hat{f}(\pi)$ respects the universal relations of $M^{\otimes_B n}$, thus it is well-defined as a bimodule map

$$\hat{f}(\pi) : M^{\otimes_B n} \to B.$$

We will denote the set of all multiplicative functions of the above form by $\mathbf{I}(M, B)$.

2.1.2. Example.

$$\begin{aligned}\hat{f}\big(\{(1,6,8),(2,5),(3,4),(7),(9)\}\big)&[a_1 \otimes \cdots \otimes a_9] = \\ &= f^{(3)}\Big(a_1 f^{(2)}\big(a_2 f^{(2)}(a_3 \otimes a_4) \otimes a_5\big) \otimes a_6 f^{(1)}(a_7) \otimes a_8\Big) f^{(1)}(a_9)\end{aligned}$$

2.1.3. Remarks. 1) Note that, although writing down the definition of $\hat{f}$ is a little bit cumbersome, the idea is quite simple and natural, namely each non-crossing partition $\pi \in NC(n)$ corresponds in a canonical way to a bracketing of an expression of n factors, and $\hat{f}(\pi)$ is nothing else than the corresponding evaluation of such a bracketing.
2) Although the decomposition $\pi = \pi_1 \cup \mathbf{1}_V$ is in general not unique, our multiplicative function $\hat{f}$ is well-defined. The result of the recursive definition does not depend on the order of decomposing into intervals.
3) A priori, $\hat{f}(\pi)$ ($\pi \in NC(n)$) is defined on the usual tensor product $M^{\otimes n}$ and one has to check that it factors down to $M^{\otimes_B n}$, i.e. that

$$\begin{aligned}\hat{f}(\pi)[a_1 \otimes \cdots \otimes a_k b \otimes a_{k+1} \otimes \cdots \otimes a_n] &= \\ &= \hat{f}(\pi)[a_1 \otimes \cdots \otimes a_k \otimes b a_{k+1} \otimes \cdots \otimes a_n]\end{aligned}$$

for all $k = 1, \ldots, n$, $a_i \in M$, $b \in B$. This can be seen as follows: There occurs only a problem if $k \not\sim_\pi k+1$. So let us assume that $\pi = \{V_1, \ldots, V_p\}$ with $k \in V_i$ and $k+1 \in V_j$ where $i \neq j$. Then at least one of the two possibilities $V_i < k+1$ or $V_j > k$ must be fulfilled. Let us consider the first one, the proof for the second one is analogous. So assume $V_i = (v_1, \ldots, v_r)$ with $v_1 < \cdots < v_r = k < k+1$. Then, by the module property of $f^{(r)}$, we have

$$\begin{aligned}\hat{f}(\pi)[a_1 \otimes \cdots \otimes a_k b \otimes a_{k+1} \otimes \cdots \otimes a_n] &= \\ &= f^{(\ldots)}\big(\ldots f^{(r)}(a_{v_1} \otimes \cdots \otimes a_k b) \otimes a_{k+1} \ldots\big) \\ &= f^{(\ldots)}\big(\ldots f^{(r)}(a_{v_1} \otimes \cdots \otimes a_k) b \otimes a_{k+1} \ldots\big) \\ &= f^{(\ldots)}\big(\ldots f^{(r)}(a_{v_1} \otimes \cdots \otimes a_k) \otimes b a_{k+1} \ldots\big) \\ &= \hat{f}(\pi)[a_1 \otimes \cdots \otimes a_k \otimes b a_{k+1} \otimes \cdots \otimes a_n].\end{aligned}$$

4) It is clear that $\hat{f}(\pi)$ ($\pi \in NC(n)$) has also the bimodule property, i.e. for all $a_1, \dots, a_n \in M$ and $b_1, b_2 \in M$ we have

$$\hat{f}(\pi)[b_1 a_1 \otimes \cdots \otimes a_n b_2] = b_1 \hat{f}(\pi)[a_1 \otimes \cdots \otimes a_n] b_2.$$

5) In the case where B is commutative and the right and left module action of B on M coincide, i.e. if $ba = ab$ for all $b \in B$ and $a \in M$, the nesting of the different $f^{(\dots)}$'s in the definition of $\hat{f}$ looses its significance and we can write $\hat{f}(\pi)$ for $\pi = \{V_1, \dots, V_p\} \in NC(n)$ with $V_i = (v_1^i, \dots, v_{k(i)}^i)$ ($i = 1, \dots, p$) in the form

$$\hat{f}(\pi)[a_1 \otimes \cdots \otimes a_n] = f^{(k(1))}(a_{v_1^1} \otimes \cdots \otimes a_{v_{k(1)}^1}) \dots f^{(k(p))}(a_{v_1^p} \otimes \cdots \otimes a_{v_{k(p)}^p}),$$

thus coming back to the form of scalar-valued multiplicative functions $\Theta \in \mathbf{I} = \mathbf{I}(\mathbb{C}, \mathbb{C})$ as considered in Remark 1.3.5.

2.1.4. NOTATION. Given two multiplicative functions $\hat{f}_1, \hat{f}_2 \in \mathbf{I}(M, B)$ and a decomposition $S_{(n)} = S_1 \cup S_2$ for some $n \in \mathbb{N}$, we will denote by

$$\begin{aligned} \hat{f}_1 \cup \hat{f}_2 : NC(S_1, S_2) \times M^{\otimes_B n} &\to B \\ (\pi_1 \cup \pi_2, a_1 \otimes \cdots \otimes a_n) &\mapsto (\hat{f}_1 \cup \hat{f}_2)(\pi_1 \cup \pi_2)[a_1 \otimes \cdots \otimes a_n] \end{aligned}$$

the function that acts on the blocks of π_1 like $\hat{f}_1$ and on the blocks of π_2 like $\hat{f}_2$. More formally, if $V = [k, l] \in \pi_1 \cup \pi_2 \in NC(S_1, S_2)$ is an interval in $S_{(n)}$ (which must either be an interval in S_1 or an interval in S_2), then we put recursively

$$\begin{aligned} &(\hat{f}_1 \cup \hat{f}_2)(\pi_1 \cup \pi_2)[a_1 \otimes \cdots \otimes a_n] := \\ &(\hat{f}_1 \cup \hat{f}_2)\big((\pi_1 \backslash \{V\}) \cup \pi_2\big)\big[a_1 \otimes \cdots \otimes a_{k-1} f_1^{(\dots)}(a_k \otimes \cdots \otimes a_l) \otimes a_{l+1} \otimes \cdots \otimes a_n\big] \end{aligned}$$

if V is an interval in S_1 and

$$\begin{aligned} &(\hat{f}_1 \cup \hat{f}_2)(\pi_1 \cup \pi_2)[a_1 \otimes \cdots \otimes a_n] := \\ &(\hat{f}_1 \cup \hat{f}_2)\big(\pi_1 \cup (\pi_2 \backslash \{V\})\big)\big[a_1 \otimes \cdots \otimes a_{k-1} f_2^{(\dots)}(a_k \otimes \cdots \otimes a_l) \otimes a_{l+1} \otimes \cdots \otimes a_n\big] \end{aligned}$$

if V is an interval in S_2, where we have, of course, that

$$(\pi_1 \backslash \{V\}) \cup \pi_2 \in NC(S_1 \backslash V, S_2) \qquad \text{or} \qquad \pi_1 \cup (\pi_2 \backslash \{V\}) \in NC(S_1, S_2 \backslash V).$$

2.1.5. EXAMPLE. For the decomposition $S_{(9)} = (1, 2, 5, 6, 8, 9) \cup (3, 4, 7)$ the Example 2.1.2 has the following modified version:

$$\begin{aligned} &(\hat{f}_1 \cup \hat{f}_2)\big(\{(1, 6, 8), (2, 5), (9)\} \cup \{(3, 4), (7)\}\big)[a_1 \otimes \cdots \otimes a_9] = \\ &\qquad = f_1^{(3)}\Big(a_1 f_1^{(2)}\big(a_2 f_2^{(2)}(a_3 \otimes a_4) \otimes a_5\big) \otimes a_6 f_2^{(1)}(a_7) \otimes a_8\Big) f_1^{(1)}(a_9). \end{aligned}$$

2.1.6. DEFINITION. Given an operator-valued multiplicative function $\hat{f} \in \mathbf{I}(M, B)$ and a scalar-valued multiplicative function $\eta \in \mathbf{I}_2$, we define now, as a generalization of our convolution of the last chapter, the function $\hat{f} \star \eta$ on $\bigcup_{n=1}^{\infty}\big(NC(n) \times M^{\otimes_B n}\big)$ by ($n \in \mathbb{N}$, $\pi \in NC(n)$)

$$(\hat{f} \star \eta)(\pi)[a_1 \otimes \cdots \otimes a_n] := \sum_{\nu \leq \pi} \hat{f}(\nu)[a_1 \otimes \cdots \otimes a_n] \eta(\nu, \pi).$$

2.1.7. PROPOSITION. *1) For $\hat{f} \in \mathbf{I}(M,B)$ and $\eta \in \mathbf{I}_2$, the function $\hat{f} \star \eta$ is multiplicative, too, namely*

$$\hat{f} \star \eta = ((\hat{f} \star \eta)^{(n)}) \in \mathbf{I}(M,B)$$

with ($n \in \mathbb{N}$, $a_1, \dots, a_n \in M$)

$$(\hat{f} \star \eta)^{(n)}(a_1 \otimes \cdots \otimes a_n) = \sum_{\pi \in NC(n)} \hat{f}(\pi)[a_1 \otimes \cdots \otimes a_n]\eta(\pi, \mathbf{1}_n).$$

2) Our operation $\star : \mathbf{I}(M,B) \times \mathbf{I}_2 \to \mathbf{I}(M,B)$ is associative in the following sense: For $\hat{f} \in \mathbf{I}(M,B)$ and $\eta_1, \eta_2 \in \mathbf{I}_2$ we have

$$(\hat{f} \star \eta_1) \star \eta_2 = \hat{f} \star (\eta_1 \star \eta_2).$$

PROOF. 1) Note first that, by our previous remarks 3) and 4), the maps $(\hat{f} \star \eta)^{(n)}$ are defined on $M^{\otimes_B n}$ and have the bimodule property, thus they may be used for defining a multiplicative function. To see that this function is indeed $\hat{f} \star \eta$, we have to check that $\hat{f} \star \eta$ fulfills the recursive relation from our definition. Let $V \in \pi$ be an interval in $S_{(n)}$ of the form $V = [k,l]$. We put $\pi_1 := \pi \backslash \{V\}$ and have thus the decomposition $\pi = \pi_1 \cup \mathbf{1}_V$. Accordingly, we decompose each $\nu \leq \pi$ in the form $\nu = \nu_1 \cup \kappa$ with $\nu_1 \leq \pi_1$ and $\kappa \leq \mathbf{1}_V$, i.e. $\nu_1 \in NC((1,\dots,n)\backslash V)$, $\kappa \in NC(V)$. From this follows (note that $\eta(\nu,\pi) = \eta(\nu_1,\pi_1)\eta(\kappa,\mathbf{1}_V)$)

$$\begin{aligned}
(\hat{f} \star \eta)(\pi)[a_1 \otimes \cdots \otimes a_n] &= \sum_{\nu \leq \pi} \hat{f}(\nu)[a_1 \otimes \cdots \otimes a_n]\eta(\nu,\pi) \\
&= \sum_{\nu_1 \leq \pi_1} \sum_{\kappa \leq \mathbf{1}_V} \hat{f}(\nu_1 \cup \kappa)[a_1 \otimes \cdots \otimes a_n]\eta(\nu_1 \cup \kappa, \pi_1 \cup \mathbf{1}_V) \\
&= \sum_{\nu_1 \leq \pi_1} \hat{f}(\nu_1)\big[a_1 \otimes \cdots \otimes a_{k-1}\big\{\sum_{\kappa \leq \mathbf{1}_V} \hat{f}(\kappa)[a_k \otimes \cdots \otimes a_l]\eta(\kappa, \mathbf{1}_V)\big\} \\
&\qquad\qquad \otimes a_{l+1} \otimes \cdots \otimes a_n\big]\eta(\nu_1,\pi_1) \\
&= \sum_{\nu_1 \leq \pi_1} \hat{f}(\nu_1)\big[a_1 \otimes \cdots \otimes a_{k-1}(\hat{f} \star \eta)^{(l-k+1)}(a_k \otimes \cdots \otimes a_l) \\
&\qquad\qquad \otimes a_{l+1} \otimes \cdots \otimes a_n\big]\eta(\nu_1,\pi_1) \\
&= (\hat{f} \star \eta)(\pi_1)\big[a_1 \otimes \cdots \otimes a_{k-1}(\hat{f} \star \eta)^{(l-k+1)}(a_k \otimes \cdots \otimes a_l) \otimes a_{l+1} \otimes \cdots \otimes a_n\big],
\end{aligned}$$

which gives the assertion.
2) We have

$$\begin{aligned}
&\big((\hat{f} \star \eta_1) \star \eta_2\big)(\pi)[a_1 \otimes \cdots \otimes a_n] = \\
&\quad = \sum_{\nu_2 \leq \pi} (\hat{f} \star \eta_1)(\nu_2)[a_1 \otimes \cdots \otimes a_n]\eta(\nu_2,\pi) \\
&\quad = \sum_{\nu_1 \leq \nu_2 \leq \pi} \hat{f}(\nu_1)[a_1 \otimes \cdots \otimes a_n]\eta_1(\nu_1,\nu_2)\eta(\nu_2,\pi) \\
&\quad = \sum_{\nu_1 \leq \pi} \hat{f}(\nu_1)[a_1 \otimes \cdots \otimes a_n](\eta_1 \star \eta_2)(\nu_1,\pi) \\
&\quad = \big(\hat{f} \star (\eta_1 \star \eta_2)\big)(\pi)[a_1 \otimes \cdots \otimes a_n]. \quad \square
\end{aligned}$$

2.2. Connection between $\hat{f}$ and $\hat{f} \star \zeta$

The transition between moment functions and cumulant functions (or vice versa) in the next chapter will be given by convolution with the zeta function (or with the Möbius function). Hence, our main interest lies in the case where η is equal to the zeta function ζ or the Möbius function μ. These cases give us the following main results of this chapter.

2.2.1. THEOREM. *1) If $\hat{f} \in \mathbf{I}(M, B)$ is an operator-valued multiplicative function, then $\hat{f} \star \zeta$ and $\hat{f} \star \mu$ are multiplicative, too, where*

$$(\hat{f} \star \zeta)(\pi)[a_1 \otimes \cdots \otimes a_n] = \sum_{\nu \leq \pi} \hat{f}(\nu)[a_1 \otimes \cdots \otimes a_n]$$

and

$$(\hat{f} \star \mu)(\pi)[a_1 \otimes \cdots \otimes a_n] = \sum_{\nu \leq \pi} \hat{f}(\nu)[a_1 \otimes \cdots \otimes a_n]\mu(\nu, \pi).$$

More explicitly, $\hat{f} \star \zeta, \hat{f} \star \mu \in \mathbf{I}(M, B)$ are the multiplicative functions given by

$$\hat{f} \star \zeta = ((\hat{f} \star \zeta)^{(n)}), \qquad \hat{f} \star \mu = ((\hat{f} \star \mu)^{(n)}),$$

where

$$(\hat{f} \star \zeta)^{(n)}(a_1 \otimes \cdots \otimes a_n) = \sum_{\pi \in NC(n)} \hat{f}(\pi)[a_1 \otimes \cdots \otimes a_n]$$

and

$$(\hat{f} \star \mu)^{(n)}(a_1 \otimes \cdots \otimes a_n) = \sum_{\pi \in NC(n)} \hat{f}(\pi)[a_1 \otimes \cdots \otimes a_n]\mu(\pi, \mathbf{1}_n).$$

2) Furthermore, the operations $\hat{f} \mapsto \hat{f} \star \zeta$ and $\hat{f} \mapsto \hat{f} \star \mu$ are inverses of each other, i.e.

$$(\hat{f} \star \zeta) \star \mu = (\hat{f} \star \mu) \star \zeta = \hat{f}.$$

PROOF. The theorem is a special case of Prop. 2.1.7, for the second part one also has to use the definition of the Möbius function μ as the inverse of ζ and the obvious statement that one has for the delta function δ

$$\hat{f} \star \delta = \hat{f} \qquad \text{for all } \hat{f} \in \mathbf{I}(M, B). \quad \square$$

Let us take a closer look on the connection between $\hat{f} = (f^{(n)})$ and $\hat{g} := (g^{(n)}) = \hat{f} \star \zeta$. The relation between the $f^{(n)}$ and the $g^{(n)}$ can also be stated in form of a recurrence formula.

2.2.2. THEOREM. *The multiplicative functions*

$$\hat{f} = (f^{(n)}) \qquad \text{and} \qquad \hat{g} = (g^{(n)}) = \hat{f} \star \zeta$$

determine each other uniquely by the recurrence formula (where we put formally $g^{(0)}(\emptyset) := 1 \in B$)

$$\begin{aligned} g^{(n)}(a_1 \otimes \cdots \otimes a_n) = \sum_{r=0}^{n-1} \sum_{1 < i(1) < i(2) \cdots < i(r) \leq n} & f^{(r+1)}\Big(a_1 g^{(i(1)-2)}(a_2 \otimes \cdots \otimes a_{i(1)-1}) \\ & \otimes a_{i(1)} g^{(i(2)-i(1)-1)}(a_{i(1)+1} \otimes \cdots \otimes a_{i(2)-1}) \\ & \otimes a_{i(2)} \cdots \otimes a_{i(r)} g^{(n-i(r))}(a_{i(r)+1} \otimes \cdots \otimes a_n)\Big) \end{aligned}$$

for all $n \in \mathbb{N}$ and all $a_1, \ldots, a_n \in M$.

PROOF. We have

$$g^{(n)}(a_1 \otimes \cdots \otimes a_n) = (\hat{f} \star \zeta)^{(n)}(a_1 \otimes \cdots \otimes a_n) = \sum_{\pi \in NC(n)} \hat{f}(\pi)[a_1 \otimes \cdots \otimes a_n].$$

We can now uniquely decompose each $\pi \in NC(n)$ in the form

$$\pi = \mathbf{1}_V \cup \pi_1 \cup \cdots \cup \pi_{r+1} \qquad (\text{where } 0 \leq r \leq n-1),$$

where $V = (1, i(1), \ldots, i(r))$ is the first block of π and $\pi_j := \pi \cap [i(j-1)+1, i(j)-1]$ $(1 \leq j \leq r+1)$ is the restriction of π to the set $[i(j-1)+1, i(j)-1]$ with $i(0) := 1$, $i(r+1) := n+1$ and $NC(0) := \{\emptyset\}$. The non-crossing character of π ensures that each block $\neq V$ of π is contained in exactly one of the π_j and that, of course, $\pi_j \in NC(i(j) - i(j-1) - 1)$. If we use this decomposition of π and carry out the sums over $\pi_1, \ldots, \pi_{r+1}$, then the asserted formula results.
Since we can iterate the recurrence formula to express $g^{(n)}$ totally in terms of the $f^{(r)}$, the recurrence formula contains all information. $\square$

In the special case where we put $a_1 = \cdots = a_n = a$, the recurrence formula can also be encoded into a relation between the corresponding generating power series. Since we are working on an algebraic level, we have no adequate notion of convergence and our power series are to be understood as formal series, the meaning of which should be clear (comp. the proof of our statement).

2.2.3. THEOREM. *Let $\hat{f} = (f^{(n)}) \in \mathbf{I}(M, B)$ be a multiplicative function and put $\hat{g} = (g^{(n)}) = \hat{f} \star \zeta$. Define the formal power series ($a \in M$)*

$$F(a) = 1 + \sum_{n \geq 1} f^{(n)}(a^{\otimes n}) \qquad \text{and} \qquad G(a) = 1 + \sum_{n \geq 1} g^{(n)}(a^{\otimes n}).$$

Then we have the following relation

$$F(aG(a)) = G(a).$$

PROOF. Theorem 2.2.2 reduces for $a_1 = \cdots = a_n = a$ to

$$g^{(n)}(a^{\otimes n}) =$$
$$= \sum_{r=1}^{n} \sum_{\substack{k_i \geq 0 \ (i=1,\ldots,r) \\ k_1 + \cdots + k_r = n-r}} f^{(r)}\Big(ag^{(k_1)}(a^{\otimes k_1}) \otimes ag^{(k_2)}(a^{\otimes k_2}) \otimes \ldots \cdots \otimes ag^{(k_r)}(a^{\otimes k_r})\Big)$$

with $g^{(0)}(a^{\otimes 0}) = 1$. But then we have

$$\begin{aligned}
F(aG(a)) &= 1 + \sum_{r \geq 1} f^{(r)}\big((aG(a))^{\otimes r}\big) \\
&= 1 + \sum_{r \geq 1} \sum_{k_i \geq 0 \ (i=1,\ldots,r)} f^{(r)}\Big(ag^{(k_1)}(a^{\otimes k_1}) \otimes \cdots \otimes ag^{(k_r)}(a^{\otimes k_r})\Big) \\
&= 1 + \sum_{n \geq 1} \sum_{r=1}^{n} \sum_{\substack{k_i \geq 0 \ (i=1,\ldots,r) \\ k_1 + \cdots + k_r + r = n}} f^{(r)}\Big(ag^{(k_1)}(a^{\otimes k_1}) \otimes \cdots \otimes ag^{(k_r)}(a^{\otimes k_r})\Big) \\
&= 1 + \sum_{n \geq 1} g^{(n)}(a^{\otimes n}) \\
&= G(a). \quad \square
\end{aligned}$$

2.3. Special case $\mathbf{I} = \mathbf{I}(\mathbb{C}, \mathbb{C})$

To get an idea of the power of Theorem 2.2.3 we will now specialize it to the scalar-valued case $\mathbf{I} = \mathbf{I}(\mathbb{C}, \mathbb{C})$ and use this for deriving some results on the combinatorics of the lattice of non-crossing partitions. These results appeared in [Spe4].

2.3.1. THEOREM. *Let $\hat{f} = (f_n) \in \mathbf{I} = \mathbf{I}(\mathbb{C}, \mathbb{C})$ be a multiplicative function determined by a sequence $(f_n)_{n\in\mathbb{N}}$ with $f_n \in \mathbb{C}$. Define $\hat{g} = (g_n) := \hat{f} \star \zeta$ and consider the formal power series*

$$F(z) := \sum_{n=0}^{\infty} f_n z^n = 1 + \sum_{n=1}^{\infty} f_n z^n$$

$$G(z) := \sum_{n=0}^{\infty} g_n z^n = 1 + \sum_{n=1}^{\infty} g_n z^n,$$

where we put $f_0 := 1$ and $g_0 := 1$. Then we have the following identities between these two formal power series

$$F(zG(z)) = G(z) \qquad \text{and} \qquad G(\frac{z}{F(z)}) = F(z).$$

2.3.2. NOTATION. In the next corollary we use the following notation: If $P(z) = \sum_{i=0}^{\infty} r_i z^i$ is a power series in z, then we denote by $\langle z^n \rangle P(z)$ the coefficient of z^n in this series, i.e. $\langle z^n \rangle P(z) = r_n$.

2.3.3. COROLLARY. *With the notations as in Theorem 2.3.1 we have*

$$g_n = \langle z^n \rangle \frac{1}{n+1} \big(1 + \sum_{i=1}^{\infty} f_i z^i\big)^{n+1} = \langle z^n \rangle \frac{1}{n+1} \big(1 + \sum_{i=1}^{n} f_i z^i\big)^{n+1}.$$

PROOF. Define $H(z) := zG(z)$ and $K(z) := (1/z)F(z)$, then we have $H(1/K(z)) = z$ and our corollary is a direct consequence of Lagrange's inversion theorem for formal power series (see, e.g., [Com, §3.8]). $\square$

In the following we shall use Theorem 2.3.1 and Corollary 2.3.3 to derive some of the results of Kreweras [Kre] and Edelman [Ede1] on the combinatorics of the lattice of non-crossing partitions in a short and uniform way.

2.3.4. COROLLARY. *The number of non-crossing partitions in $NC(n)$ which are of class $(k_i)_{i\in\mathbb{N}}$ is equal to*

$$N\big(n, (k_i)_{i\in\mathbb{N}}\big) = \frac{n!}{\big(\prod_{i\in\mathbb{N}} k_i!\big)\big(n + 1 - \sum_{i\in\mathbb{N}} k_i\big)!}.$$

PROOF. Take the setup of our Theorem 2.3.1. Then we have according to the Def. 1.3.2 of 'class'

$$g_n = \sum_{\pi\in NC(n)} \hat{f}(\pi) = \sum_{(k_i)_{i\in\mathbb{N}}} N\big(n, (k_i)_{i\in\mathbb{N}}\big) \prod_{i=1}^{\infty} f_i^{k_i}.$$

On the other hand, the multinomial identity applied to Corollary 2.3.3 gives

$$\begin{aligned} g_n &= \langle z^n \rangle \frac{1}{n+1} \Big(\sum_{i=0}^{n} f_i z^i \Big)^{n+1} \\ &= \langle z^n \rangle \frac{1}{n+1} \sum_{\substack{k_i \geq 0 \ (i=0,\dots,n) \\ k_0+\dots+k_n=n+1}} \frac{(n+1)!}{k_0! \dots k_n!} \prod_{i=0}^{n} (f_i z^i)^{k_i}. \end{aligned}$$

Comparing the coefficients of $\prod f_i^{k_i}$ and noting that $k_0 = n+1-\sum_{i=1}^{n} k_i$ gives the assertion. □

2.3.5. Remarks. 1) In the following corollaries we shall end up with a functional equation of the form

$$zS(z)^p = S(z) - 1.$$

It is known (problem 211 in [PSz], cf. also [Kla,HiP]) that the solution $S(z)$ with $S(0) = 1$ of this equation is given by

$$S(z) = 1 + \sum_{n=1}^{\infty} {}_p c_n z^n,$$

where

$${}_p c_n := \frac{1}{n} \binom{pn}{n-1}$$

are the so-called 'generalized Catalan numbers'. In [HiP] they appeared, e.g., as the number of 'p-good paths'.

2) For $p = 2$ they reduce to the usual Catalan numbers

$$c_n = {}_2 c_n = \frac{1}{n} \binom{2n}{n-1}.$$

In this case the generating function $S(z)$ is given by

$$S(z) = \frac{1 - \sqrt{1-4z}}{2z}.$$

The Catalan numbers c_n are uniquely determined by the recurrence formula

$$c_n = \sum_{k=1}^{n} c_{k-1} c_{n-k} \qquad \text{and} \qquad c_0 := 1.$$

The first few Catalan numbers are

$$c_0 = 1, \quad c_1 = 1, \quad c_2 = 2, \quad c_3 = 5, \quad c_4 = 14, \quad c_5 = 42, \quad \dots$$

We shall see in the following corollaries that the Catalan numbers are the most important numbers for the lattice of non-crossing partitions.

2.3.6. Corollary. *The number of elements of $NC(n)$ is equal to the Catalan number c_n.*

Proof. Put $f_n = 1$ for all $n \in \mathbb{N}$ in our Theorem 2.3.1, hence $F(z) = \frac{1}{1-z}$. Then

$$g_n := \sum_{\pi \in NC(n)} 1 = \#NC(n)$$

is determined via $G(z) = \sum_{n=0}^{\infty} g_n z^n$ by

$$\frac{1}{1 - zG(z)} = G(z),$$

hence by $zG(z)^2 = G(z) - 1$. Since $G(0) = 1$, we get $g_n = {}_2c_n = c_n$. $\square$

2.3.7. Notation. In the next corollary we generalize this result to the number $Z_n(p)$ of multi-chains of length p in $NC(n)$; this is the number of p-tuples $(\pi_1, \dots, \pi_p)$ with $\pi_i \in NC(n)$ for all $i = 1, \dots, p$ and $\mathbf{0}_n \le \pi_1 \le \pi_2 \le \dots \le \pi_p \le \mathbf{1}_n$. For $p = 1$, this gives of course again the number of elements of $NC(n)$, i.e. $Z_n(1) = c_n$.

2.3.8. Corollary. *The number $Z_n(p)$ of multi-chains of length p in $NC(n)$ is*

$$Z_n(p) = {}_{p+1}c_n = \frac{1}{n}\binom{(p+1)n}{n-1}.$$

Proof. Let us denote $\zeta^{(r)} := \zeta \star \dots \star \zeta$ (r factors). Then

$$\begin{aligned}\zeta^{(p)}(\mathbf{0}_n, \mathbf{1}_n) &= \sum_{\mathbf{0}_n \le \pi_1 \le \dots \le \pi_{p-1} \le \mathbf{1}_n} \zeta(\mathbf{0}_n, \pi_1)\zeta(\pi_1, \pi_2)\dots\zeta(\pi_{p-2}, \pi_{p-1})\zeta(\pi_{p-1}, \mathbf{1}_n) \\ &= Z_n(p-1).\end{aligned}$$

If we put $G^{(p)}(z) := \sum_{n=0}^{\infty} Z_n(p-1) \cdot z^n$, then we have according to our Theorem 2.3.1 the following relations

$$G^{(p)}(zG^{(p+1)}(z)) = G^{(p+1)}(z).$$

By induction, one can easily show that this implies

$$z\big(G^{(p)}(z)\big)^p = G^{(p)}(z) - 1,$$

hence $Z_n(p-1) = {}_pc_n$. $\square$

2.3.9. Corollary. *The number of those partitions in $NC(pn)$ (for $p, n \in \mathbb{N}$) such that all blocks contain exactly p elements is equal to*

$${}_pc_n = \frac{1}{n}\binom{pn}{n-1}.$$

Proof. We put

$$\begin{aligned} g_n :&= \#\{\pi = \{V_1, \dots, V_n\} \in NC(pn) \mid |V_i| = p \quad \text{for all } i = 1, \dots, n\} \\ &= \sum_{\pi \in NC(pn)} \hat{f}(\pi), \end{aligned}$$

where

$$\hat{f}(\pi) = \begin{cases} 1, & \text{if all blocks of } \pi \text{ contain } p \text{ elements} \\ 0, & \text{otherwise} \end{cases}$$

is a multiplicative function given by a sequence $(f_n)_{n\in\mathbb{N}}$ with $f_n = \delta_{n,p}$, hence $F(z) = 1 + z^p$. Thus $G(z) := \sum_{n=0}^{\infty} g_n z^{np}$ is determined by

$$1 + z^p G(z)^p = G(z),$$

hence $S(z) := \sum_{n=0}^{\infty} g_n z^n$ fulfills $S(0) = 1$ and

$$z^p S(z^p)^p = S(z^p) - 1,$$

which implies $g_n = {}_p c_n$. □

2.3.10. Corollary. *The Möbius function $\mu \in \mathbf{I}_2$ of the incidence algebra of the lattice of non-crossing partitions is determined by*

$$\mu(\mathbf{0}_n, \mathbf{1}_n) = (-1)^{n-1} c_{n-1} \qquad \textit{for all } n \in \mathbb{N}.$$

Proof. By definition we have $\mu \star \zeta = \delta$, i.e., with the notations of our Theorem 2.3.1, $f_n = \mu(\mathbf{0}_n, \mathbf{1}_n)$ and

$$g_n = \delta(\mathbf{0}_n, \mathbf{1}_n) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n \neq 1, \end{cases}$$

thus $G(z) = 1 + z$ and $F(z)$ is determined by

$$1 + \frac{z}{F(z)} = F(z),$$

or equivalently

$$\begin{aligned} F(z) &= \frac{1 + \sqrt{1 + 4z}}{2} \\ &= 1 + z \frac{1 - \sqrt{1 - 4(-z)}}{2(-z)} \\ &= 1 + z \sum_{n=0}^{\infty} c_n (-z)^n \\ &= 1 + \sum_{n=0}^{\infty} c_n (-1)^n z^{n+1}, \end{aligned}$$

from which the assertion follows. □

2.4. Tracial multiplicative functions

In the next chapter we will also consider the special case where $\hat{f}$ and $\hat{g}$ are the cumulant and moment function, respectively, of a 'trace', a notation which makes only sense if B is commutative. On the level of our multiplicative functions in $\mathbf{I}(M, B)$ this tracial property can be stated as follows.

2.4.1. Definition. Let B be commutative. A multiplicative function $\hat{f} = (f^{(n)}) \in \mathbf{I}(M, B)$ is called *tracial*, if we have for all $n \in \mathbb{N}$ and all $a_1, \dots, a_n \in M$

$$f^{(n)}(a_1 \otimes \cdots \otimes a_n) = f^{(n)}(a_n \otimes a_1 \otimes \cdots \otimes a_{n-1}).$$

2.4.2. Lemma. *Let B be commutative and $\hat{f} = (f^{(n)}) \in \mathbf{I}(M,B)$ be tracial.*
1) We have for all $n \in \mathbb{N}$, $a_1,\dots,a_n \in M$ and $b_0,b_1,\dots,b_n \in B$

$$f^{(n)}(b_0a_1b_1 \otimes a_2b_2 \otimes \cdots \otimes a_nb_n) = f^{(n)}(a_1 \otimes \cdots \otimes a_n)b_0b_1\dots b_n.$$

2) If we write $\pi = \{V_1,\dots,V_p\} \in NC(n)$ with $V_i = (v_1^i,\dots,v_{k(i)}^i)$ $(i = 1,\dots,p)$, then we have for all $a_1,\dots,a_n \in M$

$$\hat{f}(\pi)[a_1 \otimes \cdots \otimes a_n] = f^{(k(1))}(a_{v_1^1} \otimes \cdots \otimes a_{v_{k(1)}^1}) \dots f^{(k(p))}(a_{v_1^p} \otimes \cdots \otimes a_{v_{k(p)}^p}).$$

3) We have for all $n \in \mathbb{N}$, all $\pi \in NC(n)$, and all $a_1,\dots,a_n \in M$

$$\hat{f}(\pi)[a_1 \otimes \cdots \otimes a_n] = \hat{f}(\vec{\pi})[a_n \otimes a_1 \otimes \cdots \otimes a_{n-1}].$$

Proof. 1) This follows directly from the bimodule and the tracial property of $\hat{f}$.
2) Let $\pi \in NC(n)$ be of the form $\pi = \pi_1 \cup \mathbf{1}_V$ with $V = [k,l]$ as in the definition 2.1.1 of a multiplicative function. Then, by part 1), we have

$$\begin{aligned}
&\hat{f}(\pi)[a_1 \otimes \cdots \otimes a_n] = \\
&\quad = \hat{f}(\pi_1)\big[a_1 \otimes \cdots \otimes a_{k-1}f^{(l-k+1)}(a_k \otimes \cdots \otimes a_l) \otimes a_{l+1} \otimes \cdots \otimes a_n\big] \\
&\quad = \hat{f}(\pi_1)[a_1 \otimes \cdots \otimes a_{k-1} \otimes a_{l+1} \otimes \cdots \otimes a_n] \cdot f^{(l-k+1)}(a_k \otimes \cdots \otimes a_l).
\end{aligned}$$

By iteration, this gives the assertion.
3) This follows by part 2) and the definition of $\vec{\pi}$. □

2.4.3. Proposition. *Let B be commutative. Consider $\hat{f} \in \mathbf{I}(M,B)$ and $\eta \in \mathbf{I}_2$. If $\hat{f}$ is tracial, then $\hat{f} \star \eta$ is tracial, too.*

Proof. Since $\vec{\mathbf{1}}_n = \mathbf{1}_n$, we obtain

$$\begin{aligned}
(\hat{f} \star \eta)^{(n)}(a_1 \otimes \cdots \otimes a_n) &= (\hat{f} \star \eta)(\mathbf{1}_n)[a_1 \otimes \cdots \otimes a_n] \\
&= (\hat{f} \star \eta)(\vec{\mathbf{1}}_n)[a_n \otimes a_1 \otimes \cdots \otimes a_{n-1}] \\
&= (\hat{f} \star \eta)(\mathbf{1}_n)[a_n \otimes a_1 \otimes \cdots \otimes a_{n-1}] \\
&= (\hat{f} \star \eta)^{(n)}(a_n \otimes a_1 \otimes \cdots \otimes a_{n-1}). \quad \square
\end{aligned}$$

2.5. Product and cluster property

In our forthcoming considerations in chapter 5 it will be essential that a certain factorization property for $\hat{g}$ translates into a quite simple property for $\hat{f} = \hat{g} \star \zeta$, the so called cluster property.

2.5.1. Definition. Let $M_1, M_2 \subset M$ be two fixed subsets of M.
1) A multiplicative function $\hat{g} = (g^{(n)}) \in \mathbf{I}(M,B)$ has the *product property* with respect to (M_1,M_2), if we have for all $n,k,l \in \mathbb{N}$ with $1 \le k \le l \le n$ and $l-k < n-1$

$$\begin{aligned}
&g^{(n)}(a_1 \otimes \cdots \otimes a_k \otimes \cdots \otimes a_l \otimes \cdots \otimes a_n) = \\
&\quad = g^{(n-l+k-1)}\big(a_1 \otimes \cdots \otimes a_{k-1}g^{(l-k+1)}(a_k \otimes \cdots \otimes a_l) \otimes a_{l+1} \otimes \cdots \otimes a_n\big)
\end{aligned}$$

for all $a_1, \dots, a_n \in M$ with

$$a_1, \dots, a_{k-1}, a_{l+1}, \dots, a_n \in M_1 \qquad \text{and} \qquad a_k, \dots, a_l \in M_2.$$

2) A multiplicative function $\hat{f} = (f^{(n)}) \in \mathbf{I}(M, B)$ has the *cluster property* with respect to (M_1, M_2), if we have for all $n, k, l \in \mathbb{N}$ with $1 \le k \le l \le n$ and $l-k < n-1$

$$f^{(n)}(a_1 \otimes \cdots \otimes a_k \otimes \cdots \otimes a_l \otimes \cdots \otimes a_n) = 0$$

for all $a_1, \dots, a_n \in M$ with

$$a_1, \dots, a_{k-1}, a_{l+1}, \dots, a_n \in M_1 \qquad \text{and} \qquad a_k, \dots, a_l \in M_2.$$

2.5.2. Remarks. 1) The requirement $l - k < n - 1$ has to be imposed in order to ensure that the set $\{a_1, \dots, a_{k-1}, a_{l+1}, \dots, a_n\} \subset M_1$ is not empty.
2) Note that we have not required $M_1 \cap M_2 = \emptyset$. This latter condition is usually fulfilled in 'physical' occurrences of the product or cluster property, but it is not necessary in our general frame. Thus we may even have $M_1 = M_2 = M$.
3) The name 'product property' becomes most evident if $l = n$, then our condition gives (with $k \ge 2$)

$$g^{(n)}(a_1 \otimes \cdots \otimes a_{k-1} \otimes a_k \otimes \cdots \otimes a_n) = g^{(k-1)}(a_1 \otimes \cdots \otimes a_{k-1}) \cdot g^{(n-k+1)}(a_k \otimes \cdots \otimes a_n)$$

for all $a_1, \dots, a_n \in M$ with

$$a_1, \dots, a_{k-1} \in M_1 \qquad \text{and} \qquad a_k, \dots, a_n \in M_2.$$

2.5.3. Proposition. *Consider multiplicative functions $\hat{f}, \hat{g} \in \mathbf{I}(M, B)$ which are related according to*

$$\hat{g} = \hat{f} \star \zeta \qquad \text{or} \qquad \hat{f} = \hat{g} \star \mu$$

and let $M_1, M_2 \subset M$ be two fixed subsets of M. Then the following two statements are equivalent.

a) *The function $\hat{g}$ has the product property with respect to (M_1, M_2).*
b) *The function $\hat{f}$ has the cluster property with respect to (M_1, M_2).*

Proof. First, assume b) to be fulfilled. Fix $n, k, l \in \mathbb{N}$ and $a_1, \dots, a_n \in M$ as in the definition of the product and cluster property. Then the cluster property of $\hat{f}$ implies that

$$\hat{f}(\pi)[a_1 \otimes \cdots \otimes a_k \otimes \cdots \otimes a_l \otimes \cdots \otimes a_n] = 0$$

for all $\pi \in NC(n)$ which couple positions of elements from M_1 with positions of elements from M_2, i.e. for all π with $\pi \notin NC(S_{(n)} \backslash [k,l], [k,l])$. For the other partitions $\pi \in NC(S_{(n)} \backslash [k,l], [k,l])$ of the form $\pi = \pi_1 \cup \pi_2$ with $\pi_1 \in NC(S_{(n)} \backslash [k,l])$ and $\pi_2 \in NC([k,l])$ we have

$$\begin{aligned} &\hat{f}(\pi_1 \cup \pi_2)[a_1 \otimes \cdots \otimes a_k \otimes \cdots \otimes a_l \otimes \cdots \otimes a_n] = \\ &\qquad = \hat{f}(\pi_1)\big[a_1 \otimes \cdots \otimes a_{k-1} \hat{f}(\pi_2)[a_k \otimes \cdots \otimes a_l] \otimes a_{l+1} \otimes \cdots \otimes a_n\big]. \end{aligned}$$

Thus

$$\begin{aligned}
&g^{(n)}(a_1 \otimes \cdots \otimes a_k \otimes \cdots \otimes a_l \otimes \cdots \otimes a_n) = \\
&= \sum_{\substack{\pi=\pi_1\cup\pi_2 \\ \in NC(S_{(n)}\setminus[k,l],[k,l])}} \hat{f}(\pi_1 \cup \pi_2)[a_1 \otimes \cdots \otimes a_k \otimes \cdots \otimes a_l \otimes \cdots \otimes a_n] \\
&= \sum_{\pi_1 \in NC(S_{(n)}\setminus[k,l])} \hat{f}(\pi_1)\Big[a_1 \otimes \cdots \otimes a_{k-1}\big(\sum_{\pi_2\in NC([k,l])} \hat{f}(\pi_2)[a_k \otimes \cdots \otimes a_l]\big) \\
&\qquad\qquad\qquad \otimes a_{l+1} \otimes \cdots \otimes a_n\Big] \\
&= g^{(n-l+k-1)}\big(a_1 \otimes \cdots \otimes a_{k-1} g^{(l-k+1)}(a_k \otimes \cdots \otimes a_l) \otimes a_{l+1} \otimes \cdots \otimes a_n\big),
\end{aligned}$$

which gives the product property for $\hat{g} = \hat{f} \star \zeta$.
Now assume a) to be true. We prove b) by induction on n. For $n = 2$, we have for $a_1 \in M_1$ and $a_2 \in M_2$

$$f^{(2)}(a_1 \otimes a_2) = g^{(2)}(a_1 \otimes a_2) - g^{(1)}(a_1) \cdot g^{(1)}(a_2) = 0,$$

since $g^{(2)}(a_1 \otimes a_2) = g^{(1)}(a_1) \cdot g^{(1)}(a_2)$ by the product property of $\hat{g}$.
Now consider $n \geq 3$ and assume b) to be true for all $n' < n$. Fix again $k, l \in \mathbb{N}$ and $a_1, \ldots, a_n \in M$ as in the definition of the product and cluster property. Then, in particular, we have as above for all $\pi \in NC(n)$ with $\pi \neq \mathbf{1}_n$

$$\hat{f}(\pi)[a_1 \otimes \cdots \otimes a_k \otimes \cdots \otimes a_l \otimes \cdots \otimes a_n] = 0 \quad \text{for } \pi \notin NC(S_{(n)}\setminus[k,l],[k,l])$$

and

$$\begin{aligned}
&\hat{f}(\pi_1 \cup \pi_2)[a_1 \otimes \cdots \otimes a_k \otimes \cdots \otimes a_l \otimes \cdots \otimes a_n] = \\
&\qquad = \hat{f}(\pi_1)\big[a_1 \otimes \cdots \otimes a_{k-1}\hat{f}(\pi_2)[a_k \otimes \cdots \otimes a_l] \otimes a_{l+1} \otimes \cdots \otimes a_n\big]
\end{aligned}$$

for $\pi = \pi_1 \cup \pi_2 \in NC(S_{(n)}\setminus[k,l],[k,l])$ with $\pi_1 \in NC(S_{(n)}\setminus[k,l])$ and $\pi_2 \in NC([k,l])$. But then (note that $\mathbf{1}_n \notin NC(S_{(n)}\setminus[k,l],[k,l])$)

$$\begin{aligned}
&g^{(n)}(a_1 \otimes \cdots \otimes a_k \otimes \cdots \otimes a_l \otimes \cdots \otimes a_n) = \\
&\quad = f^{(n)}(a_1 \otimes \cdots \otimes a_k \otimes \cdots \otimes a_l \otimes \cdots \otimes a_n) \\
&\qquad + \sum_{\substack{\pi\in NC(n) \\ \pi\neq\mathbf{1}_n}} \hat{f}(\pi)[a_1 \otimes \cdots \otimes a_k \otimes \cdots \otimes a_l \otimes \cdots \otimes a_n] \\
&\quad = f^{(n)}(a_1 \otimes \cdots \otimes a_k \otimes \cdots \otimes a_l \otimes \cdots \otimes a_n) \\
&\qquad + \sum_{\pi_1 \in NC(S_{(n)}\setminus[k,l])} \hat{f}(\pi_1)\Big[a_1 \otimes \cdots \otimes a_{k-1}\big(\sum_{\pi_2\in NC([k,l])} \hat{f}(\pi_2)[a_k \otimes \cdots \otimes a_l]\big) \\
&\qquad\qquad\qquad \otimes a_{l+1} \otimes \cdots \otimes a_n\Big] \\
&\quad = f^{(n)}(a_1 \otimes \cdots \otimes a_k \otimes \cdots \otimes a_l \otimes \cdots \otimes a_n) \\
&\qquad + g^{(n-l+k-1)}\big(a_1 \otimes \cdots \otimes a_{k-1} g^{(l-k+1)}(a_k \otimes \cdots \otimes a_l) \otimes a_{l+1} \otimes \cdots \otimes a_n\big),
\end{aligned}$$

implying, by the product property of $\hat{g}$, that

$$f^{(n)}(a_1 \otimes \cdots \otimes a_k \otimes \cdots \otimes a_l \otimes \cdots \otimes a_n) = 0. \quad \square$$

In the next chapter $\hat{f}$ and $\hat{g}$ will get the meaning of a cumulant and moment function, respectively. This means that M carries also some multiplicative structure and the $g^{(n)}$ will have the property that $\otimes$ can be replaced by multiplication wherever this makes sense. Here, we can deal with a quite simple special case of this property, namely we choose $M_1 = M$ and $M_2 = \{a\}$, where $a \in M$ behaves like a unit element.

2.5.4. Corollary. *Consider multiplicative functions $\hat{f}, \hat{g} \in \mathbf{I}(M, B)$ which are related according to*

$$\hat{g} = \hat{f} \star \zeta \qquad \text{or} \qquad \hat{f} = \hat{g} \star \mu$$

and let $a \in M$ be a distinguished element with

$$g^{(1)}(a) = f^{(1)}(a) = 1 \in B.$$

Then the following two statements are equivalent.

a) *We have*

$$g^{(n+1)}(a_1 \otimes \cdots \otimes a_p \otimes a \otimes a_{p+1} \otimes \cdots \otimes a_n) = \\ = g^{(n)}(a_1 \otimes \cdots \otimes a_p \otimes a_{p+1} \otimes \cdots \otimes a_n)$$

for all $n \geq 1$, all $p = 1, \ldots, n$, and all $a_1, \ldots, a_n \in M$.

b) *We have*

$$f^{(n+1)}(a_1 \otimes \cdots \otimes a_p \otimes a \otimes a_{p+1} \otimes \cdots \otimes a_n) = 0$$

for all $n \geq 1$, all $p = 1, \ldots, n$, and all $a_1, \ldots, a_n \in M$.

If we have, on the other extreme, $M_1 = M_2 = M$, then the product property collapses to a 'homomorphism' property and we get the following corollary.

2.5.5. Corollary. *Consider multiplicative functions $\hat{f}, \hat{g} \in \mathbf{I}(M, B)$ which are related according to*

$$\hat{g} = \hat{f} \star \zeta \qquad \text{or} \qquad \hat{f} = \hat{g} \star \mu.$$

Then the following two statements are equivalent:

a) *The function $\hat{g} = (g^{(n)})$ is a homomorphism, i.e. we have for all $n \in \mathbb{N}$ and all $a_1, \ldots, a_n \in M$*

$$g^{(n)}(a_1 \otimes \cdots \otimes a_n) = g^{(1)}(a_1) \ldots g^{(1)}(a_n).$$

b) *The function $\hat{f} = (f^{(n)})$ is only non-vanishing for $n = 1$, i.e. we have for all $n \geq 2$ and all $a_1, \ldots, a_n \in M$*

$$f^{(n)}(a_1 \otimes \cdots \otimes a_n) = 0.$$